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**Growth estimates for conformal
mappings and for positive harmonic
functions in space**

by

Thomas F. Carroll, B.Sc., M.Sc.

A thesis submitted to The Open University for the degree of
Doctor of Philosophy.

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for my parents

Abstract

This thesis falls naturally into two distinct parts. Both come under the general heading of the theory of conformal mapping though the later part incorporates work in potential theory.

We first study the growth of means of the logarithmic derivative of a univalent function in the disc. Here results have been obtained by Hayman and by Baernstein and Brown. Hayman has shown that an elementary upper bound for the growth of these means is best possible in general. Later on, Baernstein and Brown showed that the means of certain classes of monotone slit mappings, including support points of the class S of normalised univalent functions, grow no faster than those of the Koebe function up to a multiplicative constant. The question remained open for unrestricted monotone slit mappings. We settle this question by constructing a monotone slit mapping the mean of whose logarithmic derivative grows faster than that of the Koebe function.

Following this, we discuss some recent work by Burdzy on the boundary behaviour of positive harmonic functions in Lipschitz domains and applications of this work to the angular derivative problem. Burdzy obtains his results on the angular derivative by probabilistic methods. Rodin and Warschawski later gave a classical proof of part of Burdzy's main result and related his criteria for the existence of an angular derivative to criteria which they had used previously. They were, however, unable to obtain a non-probabilistic proof of the full theorem.

Using a new non-probabilistic method, we prove a theorem on the growth of positive harmonic functions vanishing near a boundary point of a Lipschitz domain. The plane case of this result and some special cases

in space were proved by Burdzy in a series of articles. He went on to prove the full result in space in a later paper with R. J. Williams. Our result enables us to give an elementary proof of the remainder of Burdzy's theorem on the angular derivative and so complements Rodin and Warschawski's work.

We complete our study of this problem by proving two further related results on the boundary behaviour of positive harmonic functions in Lipschitz domains. It is likely that the methods used will be helpful in problems of a similar nature.

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Notation

\mathcal{C} is the complex plane and $z = x + iy$ is a complex number.

$\Re z$ is the real part of a complex number z .

$\Im z$ is the imaginary part of a complex number z .

$\Delta = \{z \in \mathcal{C} : |z| < 1\}$ is the unit disc.

\mathcal{R}^d is d -dimensional Euclidean space. A point in \mathcal{R}^d is denoted by either x or P or (X, y) with $X \in \mathcal{R}^{d-1}$ and $y \in \mathcal{R}$. We frequently identify $(X, 0)$ with X .

$B(x_0, r) = \{x \in \mathcal{R}^d : |x - x_0| < r\}$ denotes a ball with centre at x_0 and of radius r .

$S(x_0, r) = \{x \in \mathcal{R}^d : |x - x_0| = r\}$ denotes a sphere with centre at x_0 and of radius r .

$H = \{(X, y) : X \in \mathcal{R}^{d-1}, y > 0\}$ is the upper half-space in \mathcal{R}^d .

$F = \{(X, 0) : X \in \mathcal{R}^{d-1}\}$ is the boundary of H .

$d\sigma$ denotes $(d - 1)$ -dimensional Lebesgue measure in \mathcal{R}^d .

c_d represents the $(d - 1)$ -dimensional Lebesgue surface measure of the unit ball in \mathcal{R}^d .

Symbols of the form $K_i, i = 0, 1, 2, \dots$ are constants whose values may depend on the functions in question or on the dimension d but are otherwise fixed. When the value of such a constant is irrelevant, we denote it by c or C . The values of c and C may vary from one occurrence to the next.

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Chapter 1

Introduction

1.1 An overview

A fundamental topic in the theory of functions of one complex variable is that of conformal mapping. Here a one-to-one analytic function $f(z)$ maps the unit disc Δ onto a simply connected domain D . Questions which have received attention include coefficient estimates, as in the recently settled Bieberbach conjecture, growth results such as Baernstein's result on integral means, boundary distortion and so on. In the present work, we focus attention on two problems in the theory of conformal mapping, one concerning the growth of the logarithmic derivative of $f(z)$, the other concerning the angular derivative problem. Though at first sight these questions seem unrelated, a key ingredient in both methods of approach is finding growth estimates in strip-like domains. Thus the questions addressed here share a common background, a description of which is given in Section 1.2.

Once the introductory material has been disposed of in Chapter 1, we describe in Chapter 2 the context of our first problem on the logarithmic derivative of a univalent function. We consider the growth of the mean

$$I_2\left(r, \frac{f'}{f}\right) = \int_0^{2\pi} \left| \frac{r f'(re^{i\theta})}{f(re^{i\theta})} \right|^2 d\theta$$

where f is a univalent function in the unit disc Δ . An elementary bound for this mean is

$$I_2\left(r, \frac{f'}{f}\right) = O\left(\frac{1}{1-r} \log \frac{1}{1-r}\right)$$

and we outline a construction due to W. K. Hayman which shows that this bound is best possible in general.

Further progress was made by A. Baernstein and J. E. Brown who considered functions mapping onto the complement of a monotone slit, rather than a general univalent function. By a monotone slit one means a path from some finite point to infinity which intersects each circle centred on the origin at most once. They in fact require that the slit does not turn back too much on itself in a sense that is made precise later. Under these assumptions they prove that

$$I_2\left(r, \frac{f'}{f}\right) = O\left(\frac{1}{1-r}\right)$$

and so the growth of its logarithmic derivative is comparable to that of the Koebe function. The mappings they consider include support points of the class \mathcal{S} . Extreme points of \mathcal{S} are also monotone slit mappings but it is not known whether they are sufficiently well-behaved to belong to the class of functions considered by Baernstein and Brown. In any case, we construct a monotone slit mapping for which

$$I_2\left(r, \frac{f'}{f}\right) \neq o\left(\frac{1}{1-r} \log \log \frac{1}{1-r}\right)$$

so that Baernstein and Brown's result fails for a general monotone slit mapping. Chapter 3 is devoted to the construction of this example.

In Chapter 4 we turn attention to a rather different topic involving work of Krzysztof Burdzy on the growth of positive harmonic functions near a boundary point of a Lipschitz domain and on the angular derivative problem.

In proving his results Burdzy makes heavy use of probability theory and, in particular, of a recent theory of Brownian excursion laws. In a series of articles, he obtains results on such excursion laws for a half-space in \mathcal{R}^d . He then uses these results to prove theorems on the growth of positive harmonic functions near a Lipschitz boundary point and to obtain an analogous partial solution of the angular derivative problem.

Since the angular derivative problem is a classical problem in complex analysis and makes no mention of probability theory, it seems desirable to

have a classical proof of Burdzy's result. Some work in this direction is due to B. Rodin and S. E. Warschawski who showed the equivalence of Burdzy's criterion for the existence of an angular derivative to previous criteria due to themselves and to Oikawa. They also supplied a proof of one direction of Burdzy's main theorem using their own more standard methods, but they were unable to supply a classical proof of the complete result.

We use a new potential theoretic method to give an elementary proof of the generalisation to space of Burdzy's main result on the boundary behaviour of positive harmonic functions in Lipschitz domains. A probabilistic proof of this generalisation appears in a later paper by Burdzy. Our proof bypasses all of the probability theory Burdzy requires and easily yields a classical proof of Burdzy's result, fulfilling Rodin and Warschawski's hope.

In Chapter 4 we give a thorough exposition of Burdzy's work leading up to his result on the angular derivative. We also discuss relevant classical results which help to put the present work in context and we describe our main result, Theorem 4.1, in full.

Chapter 5 is devoted to the proof of Theorem 4.1 and, as such, develops our new method in detail.

We conclude, in Chapter 6, with some complementary results. The proof of Theorem 6.1 employs a now standard technique which has been used by S. E. Warschawski and by P. J. Rippon and which involves the Carleman method and results of Friedland and Hayman and of Sperner. The proof of Theorem 6.2 is elementary, though it is possible that the method used would be useful in proving other results of this type.

Our introductory chapter now continues with an exposition of material which forms the background to the thesis.

1.2 Some background material

1.2.1 The hyperbolic metric

In complex analysis, the natural distance function on the unit disc is not the usual Euclidean metric but the hyperbolic metric. This metric is chosen

so that the analytic homeomorphisms of the disc are isometries. The form of one-to-one conformal maps $w(z)$ of the disc onto itself is well known. In fact, suppose that α and β are in the disc and that $w(\alpha) = \beta$. Then, for z in Δ ,

$$\frac{w(z) - \beta}{1 - \bar{\beta}w(z)} = e^{i\phi} \frac{z - \alpha}{1 - \bar{\alpha}z}, \quad (1.2.1)$$

where $0 \leq \phi < 2\pi$. Conversely, any $w(z)$ arising from (1.2.1) is a one-to-one conformal map of the unit disc onto itself.

Let us now examine the distortion of length by $w(z)$ near α . If we let z approach α then $w(z)$ approaches β and (1.2.1) then gives

$$\frac{|dw|}{1 - |w|^2} = \frac{|dz|}{1 - |z|^2}. \quad (1.2.2)$$

Thus, if we take

$$\frac{|dz|}{1 - |z|^2} \quad (1.2.3)$$

as the element of length for a metric on the unit disc, the one-to-one conformal maps from Δ onto itself are isometries. This metric is called the hyperbolic metric.

The geodesic between the origin and a point z is along the Euclidean straight line from 0 to z . To see this, we may assume that $z = r$ is positive and we suppose that γ is a path joining 0 to r parameterised by arclength ds . Let $l_E(\gamma)$ be the Euclidean length of γ and $l(\gamma)$ be its length in the hyperbolic metric. Then $|\gamma'(s)| = 1$ for $0 \leq s \leq l_E(\gamma)$ and so

$$\begin{aligned} l(\gamma) &= \int_{\gamma} \frac{|dz|}{1 - |z|^2} \\ &\geq \int_{\gamma} \frac{|d(\Re z)|}{1 - (\Re z)^2} \\ &\geq \int_0^r \frac{d\rho}{1 - \rho^2} \\ &= \frac{1}{2} \log \frac{1+r}{1-r} \\ &= l([0, r]). \end{aligned}$$

If we write $d(0, r; \Delta)$ for the hyperbolic distance between 0 and r in Δ , then

$$d(0, r; \Delta) = \frac{1}{2} \log \frac{1+r}{1-r}. \quad (1.2.4)$$

This enables us to determine the hyperbolic distance between any two points a and b in Δ . Write

$$\phi_a(z) = \frac{z - a}{1 - \bar{a}z}.$$

Because $\phi_a(z)$ is an isometry,

$$\begin{aligned} d(a, b; \Delta) &= d(0, \phi_a(b); \Delta) \\ &= \frac{1}{2} \log \frac{1 + \delta(a, b)}{1 - \delta(a, b)}, \end{aligned}$$

where

$$\delta(a, b) = \left| \frac{b - a}{1 - \bar{a}b} \right|.$$

We may transfer all of the above theory from the disc to a general simply connected domain D , except \mathcal{C} , via the conformal mapping guaranteed by the Riemann mapping theorem. Suppose that $w_1(\zeta)$ and $w_2(\zeta)$ are one-to-one conformal maps from D onto Δ . Then

$$w_1(\zeta) = \phi(w_2(\zeta))$$

where $\phi(z)$ is one-to-one and conformal from Δ onto itself. Thus, by (1.2.2),

$$\begin{aligned} \frac{|w'_1(\zeta)|}{1 - |w_1(\zeta)|^2} &= \frac{|\phi'(w_2(\zeta))|}{1 - |\phi(w_2(\zeta))|^2} |w'_2(\zeta)| \\ &= \frac{|w'_2(\zeta)|}{1 - |w_2(\zeta)|^2}, \end{aligned}$$

so that

$$\frac{|w'(\zeta)|}{1 - |w(\zeta)|^2}$$

is independent of the choice of univalent map w from D onto Δ . We take

$$ds = \frac{|w'(\zeta)|}{1 - |w(\zeta)|^2} |d\zeta| \quad (1.2.5)$$

as the well-defined element of length in D . The resulting metric is called the Poincaré metric on D . The length of a curve $\gamma(t)$ on $[0, 1]$ in D is then

$$\begin{aligned} l(\gamma) &= \int_0^1 \frac{|\gamma'(t)| |w'(\gamma(t))|}{1 - |w(\gamma(t))|^2} dt \\ &= \int_0^1 \frac{|(w \circ \gamma)'(t)|}{1 - |(w \circ \gamma)(t)|^2} dt \\ &= l(w(\gamma)). \end{aligned}$$

Thus the length of a path in D is the length of its image in Δ under any w . Thus distance, geodesics etc. in D and in Δ all correspond under w .

An example In the case of the half-plane

$$D = \{\zeta : \Im \zeta > 0\}$$

we may take

$$w(\zeta) = \frac{\zeta - i}{\zeta + i}.$$

Thus

$$\begin{aligned} \frac{|w'(\zeta)|}{1 - |w(\zeta)|^2} &= \frac{|(\zeta + i) - (\zeta - i)|}{|\zeta + i|^2} \cdot \frac{|\zeta + i|^2}{|\zeta + i|^2 - |\zeta - i|^2} \\ &= \frac{2}{-i\zeta + i\bar{\zeta} - (i\zeta - i\bar{\zeta})} \\ &= \frac{2}{2i(\bar{\zeta} - \zeta)} \\ &= \frac{1}{2\Im \zeta}. \end{aligned}$$

The above calculation shows that in the upper half-plane

$$ds = \frac{|d\zeta|}{2\Im \zeta}.$$

1.2.2 The Green's function

We now turn our attention to $\mathcal{R}^d, d \geq 2$, and the Green's function. Let D be any domain in \mathcal{R}^d and let x_0 be a point in D . Following Hayman, ([22], page 249), we say that $g(x, x_0; D)$ is the Green's function of D with pole at x_0 if $g(x, x_0; D)$ has the following properties:

1. g is harmonic in D except at the point $x = x_0$;
2. if P is any boundary point of D , apart from a polar set E , then

$$g(x, x_0; D) \rightarrow 0 \text{ as } x \rightarrow P \text{ from inside } D$$

and if P is a point of E , $g(x, x_0; D)$ remains bounded as $x \rightarrow P$ from inside D ;

3. $g + \log |x - x_0|$ remains harmonic at $x = x_0$ if $d = 2$,

$g - |x - x_0|^{2-d}$ remains harmonic at $x = x_0$ if $d > 2$.

Polar sets are countable unions of compact sets of capacity zero. In fact, Hayman notes that it has been shown by Bouligand that the exceptional set E is precisely the set of irregular boundary points. For the definition of an irregular boundary point see [22], page 58. None of the domains we consider have irregular boundary points. One way to see this is that associated with each point P of ∂D there is an arc if $d = 2$ or a cone if $d > 2$ in the complement of D ending at P (see [22] Theorem 2.11). The existence of the Green's function is guaranteed by the following theorem which appears as Theorem 5.24 in [22].

Theorem 1.A *If D is any domain in \mathcal{R}^d whose boundary is not polar, then the Green's function of D exists and is unique.*

1.2.3 Hyperbolic distance and the Green's function

There is a simple relationship between hyperbolic distance and the Green's function in a simply connected plane domain D . We can write down the Green's function for the unit disc, $g(0, z; \Delta)$, quite simply. It is

$$g(0, z; \Delta) = \log \frac{1}{|z|}.$$

Thus

$$|z| = e^{-g(0, z; \Delta)}$$

whence, by (1.2.4),

$$d(0, z; \Delta) = \frac{1}{2} \log \frac{1 + e^{-g(0, z; \Delta)}}{1 - e^{-g(0, z; \Delta)}}. \quad (1.2.6)$$

It is a consequence of the conformal invariance of both hyperbolic distance and the Green's function that (1.2.6) holds for general points a and b in the unit disc and hence also in any simply connected domain D . In general, then,

$$d(a, b; D) = \frac{1}{2} \log \frac{1 + e^{-g(a, b; D)}}{1 - e^{-g(a, b; D)}}. \quad (1.2.7)$$

1.2.4 The Poisson integral formula

We will need the forms of the Poisson integral formula for harmonic functions in a ball, in a half-space and in a strip. In each case, the function $u(x)$ is assumed to be harmonic and bounded in the domain in question and continuous onto the boundary. For a ball $B(0, r)$ we have

$$u(x) = \frac{1}{c_d} \int_{S(0, r)} \frac{r^2 - |x|^2}{r|x - \zeta|^d} u(\zeta) d\sigma(\zeta). \quad (1.2.8)$$

In the case of the half-space H in \mathcal{R}^d i.e. $H = \{(X, y) : X \in \mathcal{R}^{d-1}, y > 0\}$,

$$u(x) = u(X, y) = \frac{2}{c_d} \int_F \frac{yu(T)}{(|X - T|^2 + y^2)^{\frac{d}{2}}} dT. \quad (1.2.9)$$

Finally, when the domain is the strip $\{z = x + iy : 0 < y < \pi\}$ in the complex plane,

$$u(z) = \frac{e^x \sin y}{\pi} \left\{ \int_{-\infty}^{\infty} \frac{e^\xi}{|e^\xi - e^z|^2} u(\xi) d\xi + \int_{-\infty}^{\infty} \frac{e^\xi}{|e^\xi + e^z|^2} u(\xi + i\pi) d\xi \right\}. \quad (1.2.10)$$

1.2.5 The maximum principle

The maximum principle tells us that subharmonic functions are dominated by their boundary values. We quote [22], Theorem 2.3.

Theorem 1.B *Suppose that $u(x)$ is subharmonic in a domain D of \mathcal{R}^d and that, if ζ is any boundary point of D and ϵ is positive, we can find a neighbourhood N of ζ such that*

$$u(x) < \epsilon \text{ in } N \cap D.$$

Then $u(x) < 0$ in D or $u(x)$ is constantly 0. If D is unbounded we consider $\zeta = \infty$ to be a boundary point of D and assume that $u(x) < \epsilon$ in $N \cap D$ when N is the exterior of some ball, $B(0, R)$.

1.2.6 The Carathéodory kernel theorem

Suppose that $\{f_n\}$ is a sequence of one-to-one conformal maps where f_n maps Δ onto a simply connected domain D_n . It is reasonable to think that if we had an appropriate definition of domain convergence and the domains D_n 'converged' to a domain D , then the maps f_n should also converge to the one-to-one conformal map from Δ to D . Carathéodory proved a result of this type in [9].

The most natural type of convergence for analytic functions in the unit disc is that of uniform convergence on compact subsets. Such convergence preserves analyticity and univalence.

The type of domain convergence appropriate in this situation is kernel convergence. Suppose that $\{D_n\}$ is a sequence of domains all of which contain a fixed point, say the origin. The kernel of the sequence $\{D_n\}$ is defined as follows. Let Ω be the set of all points in the plane which lie in all but finitely many of the domains D_n . By assumption, 0 is in Ω . If no neighbourhood of 0 lies in Ω then we put $\ker\{D_n\} = \{0\}$. Otherwise we take $\ker\{D_n\}$ to be the component of the interior of Ω containing 0 . In this case, it is easy to see that $\ker\{D_n\}$ is a simply connected domain since any closed Jordan path in $\ker\{D_n\}$, being compact, lies in D_n for large enough n and so is homotopic to a point. Finally, we say that $\{D_n\}$ converges to its kernel, $\ker\{D_n\}$, if any subsequence of $\{D_n\}$ has the same kernel.

With this notion of kernel convergence, Carathéodory proved the theorem which follows.

Theorem 1.C *Let $\{D_n\}$ be a sequence of simply connected domains of which all contain 0 but none of which is the entire plane. Let f_n map the unit disc Δ one-to-one and conformally onto D_n , normalised so that $f_n(0) = 0$ and $f'_n(0)$ is positive. Let D be the kernel of $\{D_n\}$. Then f_n converges to f uniformly on compact subsets of Δ if and only if D_n converges to D and D is not the entire plane. In the case of convergence there are two possibilities. If $D = \{0\}$, then $f = 0$. If $D \neq \{0\}$, then D is a simply connected domain, f maps Δ conformally onto D and the inverse functions f_n^{-1} converge to f^{-1} uniformly on each compact subset of D .*

A clear account of the Carathéodory kernel theorem is presented in [14], Chapter 3.

1.2.7 Characteristic constants of sets on the sphere

We begin by defining the characteristic constant $\alpha(E)$ of a measurable set E on the unit sphere.

Suppose that E is open. Then let $\mathcal{F}(E)$ be the class of Lipschitz functions f on the unit sphere which are nonnegative, not identically zero and which vanish outside E . Let

$$\lambda(E) = \inf_{f \in \mathcal{F}(E)} \frac{\int |\text{grad} f|^2 d\sigma}{\int |f|^2 d\sigma}.$$

Then the characteristic constant $\alpha(E)$ of E is defined to be the positive solution of the equation

$$\alpha(\alpha + d - 2) = \lambda.$$

If E is a compact set on the sphere we define

$$\alpha(E) = \sup\{\alpha(D) : E \subset D \text{ and } D \text{ is open}\}.$$

If E is a general measurable set on the sphere we define

$$\alpha(E) = \inf\{\alpha(F) : F \subset E \text{ and } F \text{ is compact}\}.$$

For sets E on $S(0, r)$, the sphere of radius r , we put

$$\alpha(E) = \alpha(\hat{E}),$$

where \hat{E} is the projection of E onto the unit sphere, i.e.

$$\hat{E} = \left\{ \frac{x}{r} : x \in E \right\}.$$

We need the following two results on characteristic constants. The first is due to Sperner [33]. By a spherical cap on the unit sphere we mean a set of the form

$$\{(X, y) \in S(0, 1) : y > c\}$$

for some c with $-1 \leq c < 1$.

Theorem 1.D *Among all the sets E with given $(d-1)$ -dimensional surface area on the unit sphere in \mathcal{R}^d a spherical cap has the smallest characteristic constant.*

Our second result is due to Friedland and Hayman. They prove ([18], Theorem 3)

Theorem 1.E *If E is a spherical cap of $(d-1)$ -dimensional surface area $c_d S$ on the unit sphere in \mathcal{R}^d then*

$$\alpha(E) \geq \begin{cases} \frac{1}{2} \log \left(\frac{1}{4S} \right) + \frac{3}{2}, & 0 < S \leq \frac{1}{4}, \\ 2(1-S), & \frac{1}{4} \leq S < 1. \end{cases} \quad (1.2.11)$$

1.2.8 Carleman means and Huber's inequality

Suppose that $u(x)$ is subharmonic and nonnegative in $B(0, r_0)$. The quantity

$$m(r) = \left\{ \frac{1}{c_d r^{d-1}} \int_{S(0,r)} u(x)^2 d\sigma(x) \right\}^{\frac{1}{2}}, \quad 0 < r < r_0, \quad (1.2.12)$$

is called the Carleman mean of u . It was used by Carleman to give a proof of the Denjoy conjecture on asymptotic values.

Let $D(r)$ be the intersection of the set $u(x) > 0$ and $S(0, r)$. Then we let $\alpha(r)$ be the characteristic constant of $D(r)$.

The following convexity theorem is the key to Carleman's method. In its higher dimensional form it is due to Huber [23].

Theorem 1.F *Suppose that $u(x)$ is a nonnegative and subharmonic function in $B(0, r_0)$ and that $D(r)$ is nonempty when $0 < r < r_0$. Then*

$$r \frac{d}{dr} \{ \log A(r) \} \geq 2\alpha(r) + d - 2, \quad (1.2.13)$$

where

$$A(r) = r \frac{d}{dr} \left(m(r)^2 r^{d-2} \right). \quad (1.2.14)$$

Theorem 1.F relates the Carleman mean to the characteristic constant of $D(r)$ and using Friedland and Hayman's result we can then estimate this characteristic constant in terms of the area of $D(r)$. This technique is used in Lemma 5.4 and in Theorem 6.1.

Lastly, note that we can relate $u(0, r/2)$ and $m(r)$ as follows. We put (see (1.2.8))

$$h_r(x) = \frac{1}{c_d} \int_{S(0,r)} \frac{r^2 - |x|^2}{r|x - \zeta|^d} u(\zeta) d\sigma(\zeta).$$

Then

$$u(0, \frac{r}{2}) \leq h_r(0, \frac{r}{2}) \leq \frac{3 \cdot 2^{d-2}}{c_d r^{d-1}} \int_{D(r)} u(\zeta) d\sigma(\zeta) \leq 3 \cdot 2^{d-2} m(r) \quad (1.2.15)$$

by the Schwarz inequality.

Chapter 2

Background on the monotone slit mapping example

2.1 Baernstein's Theorem

Now that the general background material has been presented we introduce the problem on the growth of the logarithmic derivative of a univalent function.

Here S is the class of functions analytic and univalent in the unit disc for which $f(0) = 0$ and $f'(0) = 1$. A special function in S is the Koebe function,

$$k(z) = \frac{z}{(1-z)^2},$$

whose image domain is the whole complex plane slit along the negative real axis from $-\frac{1}{4}$ to $-\infty$. The function $k(z)$ is extremal for many problems. The most famous is the Bieberbach conjecture solved in 1985 by L. deBranges. A problem of comparable infamy concerned integral means. If f is in S , r is in $(0, 1)$ and $0 < p < \infty$ we put

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}.$$

That is $M_p(r, f)$ is the L_p mean of $f_r(e^{i\theta}) = f(re^{i\theta})$. When $p = \infty$ we write

$$M_\infty(r, f) = \max\{|f(re^{i\theta})| : 0 \leq \theta < 2\pi\}$$

and the basic growth theorem, ([14], Theorem 2.6), says that

$$M_\infty(r, f) \leq M_\infty(r, k) = \frac{r}{(1-r)^2}.$$

In 1925, Littlewood proved

$$M_1(r, f) \leq \frac{r}{1-r},$$

whereas

$$M_1(r, k) = \frac{r}{1-r^2}.$$

The correct bound for $M_p(r, f)$ remained unproved until Baernstein's Acta paper [2] of 1973, though it had long been conjectured that $k(z)$ would dominate p^{th} -means also. Baernstein proved more.

Theorem 2.A *Let $\phi(x)$ be a convex, nondecreasing function on \mathcal{R} . Then, for each f in \mathcal{S} and $0 < r < 1$,*

$$\int_0^{2\pi} \phi(\log |f(re^{i\theta})|) d\theta \leq \int_0^{2\pi} \phi(\log |k(re^{i\theta})|) d\theta. \quad (2.1.1)$$

If ϕ is strictly convex, then equality holds for some r only if f is a rotation of k .

The choice of $\phi(x) = e^{px}$, $0 < p < \infty$, yields that for all f in \mathcal{S} ,

$$M_p(r, f) \leq M_p(r, k).$$

The proof of Theorem 2.A is based on Baernstein's star function which has proved to be a powerful tool in tackling other problems in function theory.

2.2 Means of the logarithmic derivative

2.2.1 Means and coefficients

Attention was then focused on integral means of derivatives of functions in \mathcal{S} . The progress made on the logarithmic derivative is of particular interest to us in the present context. If f is in \mathcal{S} , we write

$$I_2\left(r, \frac{f'}{f}\right) = \int_0^{2\pi} \left| \frac{rf'(re^{i\theta})}{f(re^{i\theta})} \right|^2 d\theta.$$

Upper bounds on $I_2(r, f'/f)$ of the form $O(1/(1-r))$ have consequences for both the boundary behaviour of the logarithmic function

$$g(z) = \log \frac{f(z)}{z}$$

and for its coefficients.

A few definitions are required to make the latter statement more precise. If f is in L^p , where $p \geq 1$, we define its integral modulus of continuity by

$$\omega_p(t) = \omega_p(t; f) = \sup_{0 \leq h < t} \left\{ \int_0^{2\pi} |f(re^{i(\theta+h)}) - f(re^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}.$$

A function f in H^p , $p \geq 1$, is said to belong to the smoothness class Λ_α^p where $0 < \alpha \leq 1$, if its boundary function, which is in L^p , has integral modulus of continuity

$$\omega_p(t) = O(t^\alpha).$$

A necessary and sufficient condition for f to belong to Λ_α^p if $1 \leq p < \infty$ and $0 < \alpha \leq 1$ is that

$$\|f'\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} = O \left(\frac{1}{(1-r)^{1-\alpha}} \right).$$

In particular, f is in $\Lambda_{\frac{1}{2}}^2$ if and only if

$$I_2(r, f') = \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta = O \left(\frac{1}{1-r} \right).$$

The following inclusions are known

$$H^\infty \subset \Lambda_{\frac{1}{2}}^2 \subset \text{BMOA} \subset H^p \text{ for every } p < \infty.$$

An account of the above results can be found in [13] and [15].

Returning to $f(z)$ in \mathcal{S} we see that $g(z) = \log(f(z)/z)$ is in $\Lambda_{\frac{1}{2}}^2$ if and only if

$$I_2 \left(r, \frac{f'}{f} \right) = O \left(\frac{1}{1-r} \right). \quad (2.2.1)$$

The connection between means of the logarithmic derivative and coefficients of $\log(f(z)/z)$ is made precise by the following lemma from [15]. Write

$$g(z) = 2 \sum_{n=1}^{\infty} \gamma_n z^n.$$

When $f(z) = k(z)$, then $\gamma_n = \frac{1}{n}$.

Lemma 2.A *Let γ_n be the logarithmic coefficients of a function $f(z)$ in \mathcal{S} . Then*

$$I_2\left(r, \frac{f'}{f}\right) = O\left(\frac{1}{1-r}\right) \quad (2.2.2)$$

if and only if

$$\sum_{n=1}^N n^2 |\gamma_n|^2 = O(N). \quad (2.2.3)$$

Also

$$I_2\left(r, \frac{f'}{f}\right) = O\left(\frac{1}{1-r} \log \frac{1}{1-r}\right) \quad (2.2.4)$$

if and only if

$$\sum_{n=1}^N n^2 |\gamma_n|^2 = O(N \log N). \quad (2.2.5)$$

2.2.2 An upper bound

The following lemma due to P. L. Duren ([20], page 151) gives an upper bound for $I_2(r, f'/f)$.

Lemma 2.B *If $f(z)$ is in \mathcal{S} then*

$$I_2\left(r, \frac{f'}{f}\right) = O\left(\frac{1}{1-r} \log \frac{1}{1-r}\right). \quad (2.2.6)$$

Proof The method of proof is to pass to area integrals over annuli. If $0 < r < 1$ and $\frac{1}{2} \leq r_1 < r_2 < 1$, then

$$M(r) = \max\{|f(re^{i\theta})| : 0 \leq \theta < 2\pi\},$$

$$m(r) = \min\{|f(re^{i\theta})| : 0 \leq \theta < 2\pi\},$$

$$D(r_1, r_2) = f(\{r_1 < |z| < r_2\}).$$

For the annulus $\{r_1 < |z| < r_2\}$ we have,

$$\begin{aligned} A(r_1, r_2) &= \int \int_{r_1 < |z| < r_2} \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right|^2 dx dy \\ &= \int \int_{D(r_1, r_2)} \frac{\rho d\rho d\phi}{\rho^2} \end{aligned}$$

$$\begin{aligned}
&\leq 2\pi \int_{m(r_1)}^{M(r_2)} \frac{d\rho}{\rho} \\
&= 2\pi \log \frac{M(r_2)}{m(r_1)}.
\end{aligned}$$

We know from the standard growth theorem, ([14], Theorem 2.6), that

$$m(r) \geq \frac{r}{(1+r)^2} \text{ and } M(r) \leq \frac{r}{(1-r)^2}.$$

Thus since $r_1 \geq 1/2$, we have $m(r_1) \geq 1/8$ and so

$$A(r_1, r_2) \leq 2\pi \log \frac{8}{(1-r_2)^2}. \quad (2.2.7)$$

Since $I_2(r, f'/f)$ is an increasing function of r ,

$$\begin{aligned}
I_2\left(r_1, \frac{f'}{f}\right) &\leq \frac{2}{r_2^2 - r_1^2} \int_{r_1}^{r_2} I_2\left(r, \frac{f'}{f}\right) r dr \\
&\leq \frac{2r_2^2}{r_2^2 - r_1^2} A(r_1, r_2).
\end{aligned} \quad (2.2.8)$$

Choose r_2 so that

$$r_2 = \frac{1}{2}(1 + r_1).$$

Then $r_2 - r_1 = \frac{1}{2}(1 - r_1)$ and so (2.2.7) and (2.2.8) yield

$$\begin{aligned}
I_2\left(r_1, \frac{f'}{f}\right) &\leq \frac{8\pi}{(1-r_1)} \log \frac{32}{(1-r_1)^2} \\
&= O\left(\frac{1}{1-r_1} \log \frac{1}{1-r_1}\right).
\end{aligned}$$

The surprising thing is that this elementary bound is best possible. Hayman [20] has constructed an example of a function f in \mathcal{S} for which

$$I_2\left(r, \frac{f'}{f}\right) \neq o\left(\frac{1}{1-r} \log \frac{1}{1-r}\right).$$

Thus the elementary bounds (2.2.5) and (2.2.6) are best possible in the full class \mathcal{S} . It should be noted that Hayman's results relate to mean p -valent functions and the constants he obtains are independent of the position of the zeros or any other normalisation. Moreover, some of the constants are best possible.

2.2.3 Upper bounds for monotone slit mappings

In [3], Baernstein and Brown considered support points and extreme points of \mathcal{S} . The inequality (2.1.1) with $f(z)$ and $k(z)$ replaced by their derivatives does not hold in general. A computation shows that

$$k'(z) = \frac{1+z}{(1-z)^3},$$

which is in H^p for any p less than $\frac{1}{3}$. But Lohwater, Piranian and Rudin have constructed in [26] an $f(z)$ in \mathcal{S} whose derivative is in no H^p class. However, if we restrict attention to support points of \mathcal{S} then, according to Baernstein and Brown, $k(z)$ has the largest growth up to a multiplicative constant.

Support points of \mathcal{S} are known to be monotone slit mappings. That is $\mathcal{C} \setminus f(\Delta)$ is a path Γ on $[0, \infty)$ which intersects each circle with centre the origin at most once. In addition, Γ has the following properties: Γ is an analytic arc which is asymptotic to a straight line at infinity and at the finite tip $\Gamma(0)$; Γ has what is known as the ' $\pi/4$ -property', namely that the angle between the tangent to Γ at a point and the radius vector to that point does not exceed $\pi/4$ in absolute value.

It is known that extreme points of \mathcal{S} are monotone slit mappings.

Baernstein and Brown considered the class $\mathcal{M}(\lambda)$, $0 < \lambda < \pi/2$, of monotone slit mappings, (the slit need not be analytic), with the analogous property to the $\pi/4$ -property, except with $\pi/4$ replaced by λ . More precisely, they required that for every t_1 in $(0, \infty)$

$$\limsup_{t \rightarrow t_1^+} \left| \arg \frac{\Gamma(t) - \Gamma(t_1)}{\Gamma(t_1)} \right| \leq \lambda$$

and that

$$\limsup_{t \rightarrow t_1^-} \left| \arg \frac{\Gamma(t_1) - \Gamma(t)}{\Gamma(t_1)} \right| \leq \lambda.$$

The first inequality should also hold for $t_1 = 0$. They proved that if f is in $\mathcal{M}(\lambda)$ and ϕ is convex and increasing on \mathcal{R} then

$$\int_{-\pi}^{\pi} \phi \left(\pm \log \left| \frac{rf'(re^{i\theta})}{f(re^{i\theta})} \right| \right) d\theta \leq \int_{-\pi}^{\pi} \phi \left(\pm \log C(\lambda) \left| \frac{rk'(re^{i\theta})}{k(re^{i\theta})} \right| \right) d\theta, \quad (2.2.9)$$

where $C(\lambda)$ depends only on λ . In particular, if f is in $\mathcal{M}(\lambda)$, then

$$\int_{-\pi}^{\pi} \left| \frac{rf'(re^{i\theta})}{f(re^{i\theta})} \right|^2 d\theta \leq C(\lambda) \frac{1}{1-r}, \quad (2.2.10)$$

so that $g(z) = \log(f(z)/z)$ is in $\Lambda_{\frac{1}{2}}^2$ in this case. In fact, the set of such functions g is bounded in $\Lambda_{\frac{1}{2}}^2$. Whether this solves the problem in question for extreme points is uncertain, since it is not known whether an extreme point of \mathcal{S} must lie in $\mathcal{M}(\lambda)$ for some $\lambda, 0 < \lambda < \pi/2$. In any case, the results fail if f is an unrestricted monotone slit mapping as is shown by a mapping to be constructed in Chapter 3. The existence of such a mapping was suggested earlier by Hayman (see remark in [15] page 38). There is a monotone slit mapping f for which

$$I_2 \left(r, \frac{f'}{f} \right) \neq o \left(\frac{1}{1-r} \log \log \frac{1}{1-r} \right).$$

2.2.4 Hayman's example

We now outline the example given by Hayman in [20]. Hayman constructs a univalent function $g(z)$ in Δ whose image $g(\Delta)$ is contained in the strip $\{w : |\Im w| < \pi\}$ and for which

$$\int_0^{2\pi} |g'(re^{i\theta})|^2 d\theta \neq o \left(\frac{1}{1-r} \log \frac{1}{1-r} \right). \quad (2.2.11)$$

Then $f(z) = e^{g(z)}$ is the required conformal mapping.

The image of $g(z)$ is depicted in Figure 2.1.

He proves that $g(z)$ satisfies (2.2.11) by showing that there are sequences r_k and r'_k tending to 1 with $r_k < r'_k$,

$$\frac{1-r_k}{1-r'_k} < K_1 \quad (2.2.12)$$

and

$$A(r_k, r'_k) = \int_0^{2\pi} \int_{r_k}^{r'_k} |g'(re^{i\theta})|^2 r dr d\theta > K_2 \log \frac{1}{1-r_k}. \quad (2.2.13)$$

Note that $A(r_k, r'_k)$ is the area of the image under $g(z)$ of the annulus $\{z : r_k < |z| < r'_k\}$. The openings d_n are chosen so that the centres s_n of

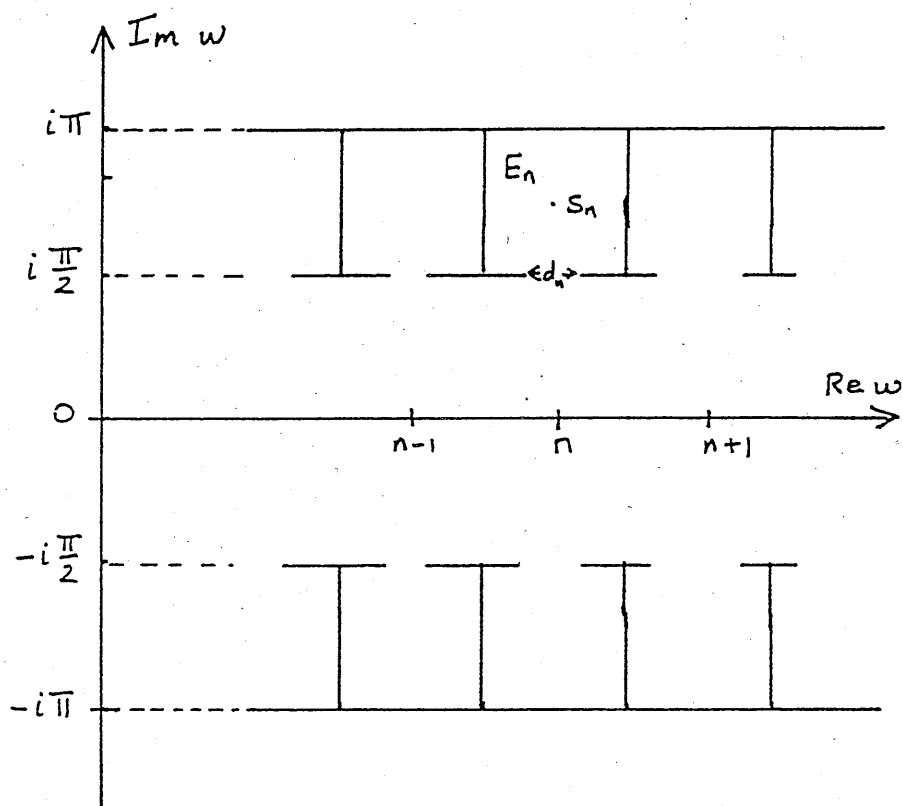


Figure 2.1 Hayman's domain

the boxes E_n for n in the range

$$2^{(k-1)^2} \leq n < 2^{k^2}, \quad (2.2.14)$$

all lie at the same hyperbolic distance from the origin up to an additive constant. In fact, for each $k > 1$ and each n in the range (2.2.14) we have

$$2^{k^2} + k - O(1) < 2d(s_n, 0; D) < 2^{k^2} + k + O(1).$$

Thus we have

$$2^{k^2} + k - O(1) < 2d(w, 0; D) < 2^{k^2} + k + O(1)$$

for all w in a box about s_n whose area is half that of E_n . The total area of these boxes in the range (2.2.14) exceeds $K2^{k^2}$ for a fixed constant K . On putting

$$\frac{1}{2} \log \frac{1+r_k}{1-r_k} = 2^{k^2} + k - O(1) \text{ and } \frac{1}{2} \log \frac{1+r'_k}{1-r'_k} = 2^{k^2} + k + O(1)$$

it follows that (2.2.12) holds. Also the Euclidean area of the image of the annulus $\{z : r_k < |z| < r'_k\}$ exceeds

$$K2^{k^2} > K_2 \log \frac{1}{1-r_k},$$

which is (2.2.13).

The above example is not a monotone slit mapping. In Chapter 3 we show how Hayman's construction can be adapted to obtain a similar example which is also a monotone slit mapping. This shows that the inequality (2.2.10), and hence (2.2.9), fails in the limiting case $\lambda = \pi/2$.

Chapter 3

A monotone slit mapping with large logarithmic derivative

3.1 Statement of the theorem and a basic lemma

We construct an example which shows that $g(z) = \log(f(z)/z)$ is not necessarily in $\Lambda_{\frac{1}{2}}^2$ if $f(z)$ is a monotone slit mapping. The example also shows that Baernstein and Leung's inequality (2.2.10), and hence (2.2.9), fails in the limiting case $\lambda = \pi/2$.

For convenience, we define a monotone slit, $\Gamma(t)$, to be a curve on $[0, \infty)$ which intersects each circle with centre the origin precisely once. Thus $\Gamma(0) = 0$.

A monotone slit mapping is a univalent function in the unit disc Δ whose image domain is precisely the complement of a monotone slit. Denote the class of monotone slit mappings by \mathcal{M} .

Theorem 3.1 *There is a function $\mathcal{F}(z)$ in \mathcal{M} for which*

$$I_2\left(r, \frac{\mathcal{F}'}{\mathcal{F}}\right) \neq o\left(\frac{1}{1-r} \log \log \frac{1}{1-r}\right).$$

Theorem 3.1 follows from the next lemma.

Lemma 3.1 *There is a function $\mathcal{G}(z)$ for which*

$$\mathcal{F}(z) \text{ is in } \mathcal{M} \text{ where } \mathcal{F}(z) = \exp(\mathcal{G}(z)) \quad (3.1.1)$$

and there are sequences r_k and r'_k which tend to 1, for which $r_k < r'_k$ and $(1 - r_k)/(1 - r'_k)$ is bounded and for which

$$A(r_k, r'_k) = \int_{r_k}^{r'_k} \int_0^{2\pi} |\mathcal{G}'(re^{i\theta})|^2 r d\theta dr > C_0 \log \log \frac{1}{1 - r'_k} \quad (3.1.2)$$

where C_0 is an absolute constant.

Mimicking an argument in [20] page 167 yields

$$I_2(r'_k, \mathcal{G}') = \int_0^{2\pi} |\mathcal{G}'(r'_k e^{i\theta})|^2 d\theta > \frac{C}{1 - r'_k} \log \log \frac{1}{1 - r'_k}.$$

In fact, since $I_2(r, \mathcal{G}')$ is an increasing function of r ,

$$\begin{aligned} I_2(r'_k, \mathcal{G}') &\geq \frac{2}{r_k'^2 - r_k^2} \int_{r_k}^{r'_k} I_2(r, \mathcal{G}') r dr \\ &= \frac{2}{r_k'^2 - r_k^2} A(r_k, r'_k) \\ &> \frac{2C_0}{r_k'^2 - r_k^2} \log \log \frac{1}{1 - r'_k} \\ &\geq \frac{C}{1 - r'_k} \log \log \frac{1}{1 - r'_k}. \end{aligned}$$

The last inequality holds because $(1 - r_k)/(1 - r'_k)$ is bounded. Theorem 3.1 now follows since

$$I_2\left(r'_k, \frac{\mathcal{F}'}{\mathcal{F}}\right) = r_k'^2 I_2(r'_k, \mathcal{G}').$$

Lemma 3.1 is proved in two stages. Firstly, an intermediate mapping, $g(z)$, from the unit disc to a symmetric domain \mathcal{D} is obtained which satisfies (3.1.2). The domain \mathcal{D} is then perturbed to a domain \mathcal{H} so that (3.1.2) remains valid for $\mathcal{G}(z) : \Delta \mapsto \mathcal{H}$ and so that (3.1.1) also holds. It is more convenient to first make the necessary estimates in the symmetric domain \mathcal{D} and then show that they remain valid in an admissible domain \mathcal{H} which is close to \mathcal{D} .

3.2 Estimates of hyperbolic distance

Suppose that $\{a_n\}_1^\infty$ is an unbounded, increasing sequence of points on the real axis for which a_1 is nonnegative.

Definition 3.1 Define a domain D corresponding to the given sequence $\{a_n\}$ by

$$D = \{z : |\Im z| < \pi, \text{ unless } \Re z = a_n \text{ for some } n, \text{ in which case } |\Im z| < \frac{\pi}{2}\}.$$

We write $2d_n = a_{n+1} - a_n$. Lastly, for each n we define

$$B_n^+ = \{z : a_n < \Re z < a_{n+1}, \frac{\pi}{2} < \Im z < \pi\}$$

and

$$B_n^- = \{z : a_n < \Re z < a_{n+1}, -\pi < \Im z < -\frac{\pi}{2}\}.$$

In the next section, a specific choice of the spacings d_n is made and it will be shown that, in this case, the conformal mapping from the unit disc to D satisfies (3.1.2). Estimates of hyperbolic distance in D are needed to do this, along the real axis in the first instance and also from the real axis to points in B_n^+ .

3.2.1 Distance along the real axis

A lower bound

We will need the following lemma which is a special case of a result on Steiner symmetrisation ([21] Chapter 5). In this special case we give an elementary proof which we were told of by P. J. Rippon. A domain D is said to be convex with respect to the imaginary axis if whenever $x + iy_1$ and $x + iy_2$ lie in D then so does the line segment joining them.

Lemma 3.2 *Let D be a simply connected domain which is symmetric about the real axis and convex with respect to the imaginary axis. Suppose also that 0 is in D . Then*

$$\max_y \{g(0, x + iy; D)\} = g(0, x; D)$$

where $g(0, z; D)$ is the Green's function for D with pole at 0.

Proof Suppose that $z_1 = x$ and that $z_2 = x + iy$, where y is positive, are in D . We wish to show that $g(0, z_1; D) \geq g(0, z_2; D)$. Write $D_{x,y}$ for the component of $D \cap \{z : \Im z > y/2\}$ containing z_2 . Then

$$u_1(z) = g(0, z; D)$$

is harmonic in $D_{x,y}$ and continuous in $\overline{D}_{x,y}$.

Write $u_2(z)$ for the function in $D_{x,y}$ given by

$$\begin{aligned} u_2(z) &= u_2(t + i\tau) \\ &= g(0, t + i(y - \tau); D). \end{aligned}$$

Since D is symmetric and convex with respect to the real axis, $u_2(z)$ is well-defined. Moreover, $u_2(z)$ is harmonic in $D_{x,y}$ if iy is not in $D_{x,y}$ and superharmonic if iy is in $D_{x,y}$, but superharmonic in any case.

Thus

$$u(z) = u_2(z) - u_1(z)$$

is superharmonic in $D_{x,y}$ and continuous in $\overline{D}_{x,y}$. By inspection, $u(z) \geq 0$ on $\partial D_{x,y}$ and so by the maximum principle $u(z) \geq 0$ in $D_{x,y}$. In particular

$$\begin{aligned} g(0, x; D) &= u_2(x + iy) \\ &\geq u_1(x + iy) = g(0, x + iy; D). \end{aligned}$$

This completes the proof of Lemma 3.2.

Let a and b be points of the real axis, $0 \leq a < b$. Theorem 1 of [19] gives a lower bound for the hyperbolic distance $d(a, b; D)$ between them.

Suppose that $\omega_n, n = 0, 1, 2, \dots$ is a sequence of complex numbers for which $|\omega_n| = r_n$ is strictly increasing and unbounded and for which

$$\omega_0 = 0, \omega_1 = -1.$$

Write δ_n for $\log(r_{n+1}/r_n)$ if $n \geq 1$, and write e_n for $\min(\delta_n, \delta_n^2)$. Then Theorem 1 of [19] runs as follows:

Theorem 3.A *If $f(\zeta)$ is regular in $|\zeta| < 1$, $|f(0)| \leq 1$ and $f(\zeta)$ never takes the values ω_n , then for $r_n < M(\rho, f) \leq r_{n+1}$ we have*

$$\log M(\rho, f) < 2 \log \frac{1+\rho}{1-\rho} + \sum_1^n e_i + 30.$$

Here $M(\rho, f) = \max\{|f(\rho e^{i\theta})| : 0 \leq \theta < 2\pi\}$.

We have

Lemma 3.3 *If $a_n < a \leq a_{n+1}$, $a_m < b \leq a_{m+1}$ and $a_{n+1} - a \leq 1$, then*

$$d(a, b; D) > \frac{b-a}{2} - \frac{1}{4} \sum_{n+1}^m e_i - 8 \quad (3.2.1)$$

where $e_i = \min(4d_i, 16d_i^2)$.

Proof We first estimate $d(0, x; D)$ where x is positive. Let $h(\zeta)$ be the conformal mapping from Δ to D for which $h(0) = 0$ and $h'(0)$ is positive. Then $h(\zeta)$ omits all points $a_n + i(\pi/2 + k\pi)$, $n = 1, 2, \dots$, and k in \mathcal{Z} . Hence

$$f(\zeta) = e^{2(h(\zeta) - a_1)}$$

omits ω_n , $n = 0, 1, \dots$, where $\omega_0 = 0$, $\omega_1 = -1$, and $\omega_n = -e^{2(a_n - a_1)}$, $n > 1$. Moreover, $|f(0)| = e^{-2a_1} \leq 1$ since $a_1 \geq 0$. Thus $f(\zeta)$ satisfies the hypotheses of Theorem 3.A. If $x = h(\rho)$ and $|\zeta| = \rho$ then $\Re h(\zeta) \leq x$. Otherwise, by Lemma 3.2 and since the Green's function for D is decreasing on $(-\infty, \infty)$,

$$g(0, h(\zeta); D) \leq g(0, \Re h(\zeta); D) < g(0, x; D)$$

which contradicts the assumption that ρ and ζ lie on a level line for the Green's function. Hence,

$$M(\rho, f) = e^{2(x - a_1)}.$$

Thus, $r_n < M(\rho, f) \leq r_{n+1}$ if and only if $a_n < x \leq a_{n+1}$.

Also, for $n \geq 1$,

$$\delta_n = \log \left| \frac{\omega_{n+1}}{\omega_n} \right| = 2(a_{n+1} - a_n) = 4d_n.$$

Theorem 3.A yields

$$2x - 2a_1 < 2 \log \frac{1+\rho}{1-\rho} + \sum_1^n e_i + 30,$$

and so, if $a_1 \leq 1$,

$$d(0, x; D) > \frac{x}{2} - \frac{1}{4} \sum_1^n e_i - 8. \quad (3.2.2)$$

Thus, if $a_n < a \leq a_{n+1}$, $a_m < b \leq a_{m+1}$ and $a_{n+1} - a \leq 1$, then

$$d(a, b; D) > \frac{b-a}{2} - \frac{1}{4} \sum_{n+1}^m e_i - 8 \quad (3.2.3)$$

where $e_i = \min(4d_i, 16d_i^2)$ which is (3.2.1).

An upper bound

This is much easier.

Lemma 3.4 *If $0 \leq a < b$, then*

$$d(a, b; D) \leq \frac{b-a}{2}. \quad (3.2.4)$$

Proof Since D contains the strip \mathcal{S} ,

$$\mathcal{S} = \{z : |\Im z| < \pi/2\},$$

it follows that

$$d(a, b; D) \leq d(a, b; \mathcal{S}) = \frac{b-a}{2}.$$

3.2.2 Distance into a box

We suppose that $d_n < \pi/4$. We write $x_n = (a_{n+1} + a_n)/2$ and let s be a point of the box B_n^+ . Estimates of the distance from s to the real axis are obtained in the next lemma, but first of all we quote [20], Lemma 6 which will be useful in the proof.

Lemma 3.A Suppose that D_0 is a simply connected domain containing the rectangle

$$R_0 = \{s : s = \sigma + \imath\tau, |\sigma| < d, -\tau_0 - d < \tau < \tau_0 + d\}$$

where τ_0 is positive. Then

$$d(-\imath\tau_0, \imath\tau_0; D_0) \leq \frac{\pi}{2d}\tau_0 + \frac{\pi}{2}.$$

Suppose further that the complement of D_0 contains the points $s = \imath\tau \pm d$ for $-\tau_0 < \tau < \tau_0$ except possibly for a set of τ having linear measure l . If $|\sigma_1| < d$ and $|\sigma_2| < d$ then

$$d(\sigma_1 - \imath\tau_0, \sigma_2 + \imath\tau_0; D_0) \geq \frac{\pi}{2d} \left(\tau_0 - \frac{l}{2} \right) - \frac{\pi}{2}.$$

We can now state

Lemma 3.5 Suppose that $s = x_n + \imath(\pi/2 + \rho)$ where ρ lies in $[d_n, \pi/2 - d_n]$. Then

$$d(x_n, s; D) \leq \frac{\pi}{4} \left(\frac{\rho}{d_n} \right) + \frac{1}{2} \log \frac{1}{d_n} + K_1 \quad (3.2.5)$$

where $K_1 = \frac{1}{2} \log 2\pi + \pi/4 + \log 7$, and

$$d(x_n, s; D) \geq \frac{\pi}{4} \left(\frac{\rho}{d_n} \right) - \frac{3\pi}{4}. \quad (3.2.6)$$

Furthermore, if x is the point on the real axis closest to s we have

$$d(x_n, s; D) - d(x, s; D) \leq \frac{1}{2} \log \left(\frac{1}{d_n} \right) + K_2 \quad (3.2.7)$$

where $K_2 = \frac{1}{2} \log 2\pi + \pi + \log 7$.

Proof Let s_n^+, s_n^- be the points $x_n + \imath(\pi/2 + d_n), x_n + \imath(\pi - d_n)/2$ respectively. By the triangle inequality,

$$d(x_n, s; D) \leq d(x_n, s_n^-; D) + d(s_n^-, s_n^+; D) + d(s_n^+, s; D). \quad (3.2.8)$$

Let B_1 be the disc $|z - x_n| < \pi/2$. Then

$$\begin{aligned} d(x_n, s_n^-; D) &\leq d(x_n, s_n^-; B_1) \\ &= \frac{1}{2} \log \frac{2\pi - d_n}{d_n} \\ &< \frac{1}{2} \log \frac{1}{d_n} + \frac{1}{2} \log 2\pi. \end{aligned} \quad (3.2.9)$$

Let B_2 be the disc $|z - x_n - i(\pi/2 + d_n/4)| < d_n$. Then,

$$\begin{aligned} d(s_n^-, s_n^+; D) &\leq d(s_n^-, s_n^+; B_2) \\ &= \log 7. \end{aligned} \quad (3.2.10)$$

D contains the rectangle

$$\{z = x + iy : a_n < x < a_{n+1} \text{ and } \frac{\pi}{2} < y < \frac{\pi}{2} + \rho + d_n\}$$

and the vertical sides of the rectangle are each part of the boundary of D .

Thus it follows from Lemma 3.A that

$$d(s_n^+, s; D) \leq \frac{\pi}{4} \left(\frac{\rho}{d_n} \right) + \frac{\pi}{4} \quad (3.2.11)$$

and, since l equals 0, we have for $-d_n < t < d_n$

$$d(s_n^+ + t, s; D) \geq \frac{\pi}{4} \left(\frac{\rho}{d_n} \right) - \frac{3\pi}{4}. \quad (3.2.12)$$

The inequality (3.2.8) and the estimates (3.2.9), (3.2.10), (3.2.11) together yield (3.2.5) with the stated value of K_1 .

The inequality (3.2.6) is no more difficult. Let γ_n be the geodesic from s to that point x of the real axis closest to s . Let Q_n be the point where γ_n meets the line $\Im z = \pi/2 + d_n$ on its way from s . Because γ_n is a geodesic, $d(x, s; D) = l(\gamma_n) > d(s, Q_n; D)$ where $l(\gamma_n)$ is the length of γ_n .

By (3.2.12),

$$d(s, Q_n; D) \geq \frac{\pi}{4} \left(\frac{\rho}{d_n} \right) - \frac{3\pi}{4}. \quad (3.2.13)$$

So we see by (3.2.5) and (3.2.13) that (3.2.7) holds. Moreover, since $d(x_n, s; D) \geq l(\gamma_n)$, (3.2.6) follows from (3.2.13) and the proof of Lemma 3.5 is complete.

3.3 The intermediate mapping

Now that these estimates of distance have been made, the next task is to make a specific choice of the numbers a_n in Definition 3.1, or equivalently of the spacings d_n , to construct the intermediate domain.

We suppose that $a_1 = 0$ and $2d_1 = 1$. For each $k > 1$, we define

$$2d_n = \frac{1}{2\sqrt{2^k - n}} \quad (3.3.1)$$

when $2^{k-1} < n < 2^k$ and, for $k \geq 1$, we define

$$2d_{2^k} = k + 1. \quad (3.3.2)$$

This defines the intermediate domain \mathcal{D} in accordance with Definition 3.1.

We need the following lemma, ([20], Lemma 4), which yields an estimate for the error in the triangle inequality for hyperbolic distance in \mathcal{D} .

Lemma 3.B *Suppose that D is a simply connected domain with a line of symmetry L . Let w_1 be a point of D not in L and let w_2 and w_3 be points of D on L . We let δ be the minimum hyperbolic distance of w_1 from L with respect to D and we put*

$$\alpha = d(w_1, w_2) - \delta.$$

Then,

$$d(w_1, w_3) \geq d(w_1, w_2) + d(w_2, w_3) - 2\alpha - \log 2.$$

We now prove

Lemma 3.6 *There are sets Ω_k in \mathcal{D} , $k = 1, 2, \dots$ and there is an unbounded, increasing sequence $\{\lambda_k\}$ such that*

$$\lambda_k - C_1 < d(0, z; \mathcal{D}) < \lambda_k + C_1$$

for large k whenever z is in Ω_k and such that the Euclidean area of Ω_k exceeds

$$C_2 \log \lambda_k$$

where C_1 and C_2 are absolute constants.

Proof We shall show that it is possible to choose a point from the boxes B_n^+, B_n^- , with $2^{k-1} < n < 2^k - K_0$ for a fixed constant K_0 and a specific k , so that these points all lie at the same distance from the origin.

Let s be a point $x_n + i(\pi/2 + \rho)$ in the n^{th} -box, B_n^+ , and in the k^{th} -block so that $2^{k-1} < n < 2^k$. By the triangle inequality,

$$\begin{aligned} d(0, s) &\leq d(0, x_n) + d(x_n, s) \\ &= d(0, x_n) + d(x_n, x_{2^{k-1}}) + d(x_n, s) - d(x_n, x_{2^{k-1}}) \\ &= d(0, x_{2^{k-1}}) + d(x_n, s) - d(x_n, x_{2^{k-1}}). \end{aligned}$$

Or,

$$d(0, s) - d(0, x_{2^{k-1}}) \leq d(x_n, s) - d(x_n, x_{2^{k-1}}). \quad (3.3.3)$$

In the other direction, Lemma 3.B gives

$$d(0, s) \geq d(0, x_n) + d(x_n, s) - 2\alpha - \log 2$$

where α is the discrepancy between the minimum distance from s to the line of symmetry of \mathcal{D} , which is the real axis, and the distance to the specific point x_n of \mathcal{R} . It follows from (3.2.7) that

$$\alpha \leq \frac{1}{2} \log \frac{1}{d_n} + K_2.$$

Thus,

$$\begin{aligned} d(0, s) &\geq d(0, x_n) + d(x_n, s) - \log \frac{1}{d_n} - 2K_2 - \log 2 \\ &= d(0, x_{2^{k-1}}) + d(x_n, s) - d(x_n, x_{2^{k-1}}) - \log \frac{1}{d_n} - K_3, \end{aligned}$$

where $K_3 = 2K_2 + \log 2$. Or,

$$d(0, s) - d(0, x_{2^{k-1}}) \geq d(x_n, s) - d(x_n, x_{2^{k-1}}) - \log \frac{1}{d_n} - K_3. \quad (3.3.4)$$

We define

$$\lambda(\rho) = d(0, s) - d(0, x_{2^{k-1}}) = d(0, x_n + i(\pi/2 + \rho)) - d(0, x_{2^{k-1}}).$$

If $\lambda(d_n)$ is negative and $\lambda(\pi/2 - d_n)$ is positive then, by continuity, $\lambda(\rho_n)$ equals 0 for some ρ_n in $(d_n, \pi/2 - d_n)$.

It follows from (3.3.3) that $\lambda(d_n)$ is negative if

$$d(x_n, s) - d(x_n, x_{2^k-1}) < 0 \quad (3.3.5)$$

when $s = x_n + \iota(\pi/2 + d_n)$, and from (3.3.4) that $\lambda(\pi/2 - d_n)$ is positive if

$$d(x_n, s) - d(x_n, x_{2^k-1}) - \log \frac{1}{d_n} - K_3 > 0 \quad (3.3.6)$$

when $s = x_n + \iota(\pi - d_n)$. Lemmas 3.3 and 3.4 yield

$$\frac{1}{2}(x_{2^k-1} - x_n) - 4 \sum_{n+1}^{2^k-1} d_i^2 - 8 \leq d(x_n, x_{2^k-1}) \leq \frac{1}{2}(x_{2^k-1} - x_n).$$

Also, for $s = x_n + \iota(\pi/2 + \rho)$, where ρ is in $[d_n, \pi/2 - d_n]$, we obtain from Lemma 3.5 that

$$d(x_n, s) \leq \frac{\pi}{4} \left(\frac{\rho}{d_n} \right) + \frac{1}{2} \log \frac{1}{d_n} + K_1$$

and that

$$d(x_n, s) \geq \frac{\pi}{4} \left(\frac{\rho}{d_n} \right) - \frac{3\pi}{4}.$$

Thus (3.3.5) holds if

$$\left[\frac{\pi}{4} \left(\frac{\rho}{d_n} \right) + \frac{1}{2} \log \frac{1}{d_n} + K_1 - \left\{ \frac{1}{2}(x_{2^k-1} - x_n) - 4 \sum_{n+1}^{2^k-1} d_i^2 - 8 \right\} \right]_{\rho=d_n} < 0$$

i.e. if

$$\frac{1}{2} \log \frac{1}{d_n} - \frac{1}{2}(x_{2^k-1} - x_n) + 4 \sum_{n+1}^{2^k-1} d_i^2 + K_4 < 0 \quad (3.3.7)$$

where $K_4 = \pi/4 + K_1 + 8$. Likewise, (3.3.6) holds if

$$\left[\frac{\pi}{4} \left(\frac{\rho}{d_n} \right) - \frac{3\pi}{4} - \frac{1}{2}(x_{2^k-1} - x_n) - \log \frac{1}{d_n} - K_3 \right]_{\rho=\pi/2-d_n} > 0$$

that is, if

$$\frac{\pi^2}{8} \left(\frac{1}{d_n} \right) - \frac{1}{2}(x_{2^k-1} - x_n) - \log \frac{1}{d_n} - K_5 > 0, \quad (3.3.8)$$

where $K_5 = \pi + K_3$. We write $n = 2^k - N$, so that $d_n = \frac{1}{4}N^{-\frac{1}{2}}$ and obtain from (3.3.1)

$$x_{2^k-1} - x_n = \frac{1}{2} \sum_1^N \frac{1}{\sqrt{i}} - \frac{1}{4} \left(1 + \frac{1}{\sqrt{N}} \right)$$

and

$$4 \sum_{n+1}^{2^k-1} d_i^2 = \frac{1}{4} \sum_1^{N-1} \frac{1}{i}.$$

We note that

$$2\sqrt{N} - 2 < \sum_1^N \frac{1}{\sqrt{i}} < 2\sqrt{N}$$

and that

$$\log N < \sum_1^N \frac{1}{i} < \log N + 1.$$

Thus (3.3.7) holds if

$$\frac{1}{2} \log(4\sqrt{N}) - \frac{1}{2}(\sqrt{N} - \frac{3}{2}) + \frac{1}{4}(\log N + 1) + K_4 < 0,$$

that is, if

$$\sqrt{N} \geq \log N + K_6.$$

Similarly, (3.3.8) holds if

$$\left(\frac{\pi^2}{2} - \frac{1}{2}\right) \sqrt{N} \geq \frac{1}{2} \log N + K_7$$

that is, if

$$\sqrt{N} \geq \frac{1}{(\pi^2 - 1)} \log N + K_8.$$

Thus (3.3.7) and (3.3.8) both hold if $N > K_0$, where K_0 is a suitable absolute constant.

Hence, for large fixed k , there are points $s_n = x_n + i(\pi/2 + \rho_n)$ in each of the boxes B_n^+, B_n^- with

$$2^{k-1} < n \leq 2^k - K_0$$

for which

$$\lambda(\rho_n) = 0.$$

In other words, the points s_n all lie at the same distance $d(0, x_{2^k-1})$ from the origin. Write $\lambda_k = d(0, x_{2^k-1})$. Then

$$\begin{aligned} \lambda_k &< \frac{1}{2} x_{2^k-1} \\ &< \frac{1}{2} \sum_{j=1}^k \left(\sum_{n=1}^{2^{j-1}} \frac{1}{2\sqrt{n}} + j + 1 \right) \end{aligned}$$

so that $\log \lambda_k \leq K_9 \log 2^k$ for large k .

Around each point s_n lies a disc of radius $d_n/2$ whose points are distant no more than $\frac{1}{2} \log 3$ from s_n . We define Ω_k to be the union of these discs and C_1 to be $\frac{1}{2} \log 3$. To complete the proof of Lemma 3.6, it remains to show that the area of Ω_k exceeds $C_2 \log \lambda_k$ for a fixed constant C_2 . We have

$$\begin{aligned} \text{Area of } \Omega_k &= \frac{\pi}{64} \sum_{K_0}^{2^{k-1}-1} \frac{1}{N} \\ &> \frac{\pi}{64} \log(2^{k-1} - 1) - \frac{\pi}{64} (1 + \log K_0) \\ &> \frac{\pi}{128} \log 2^k, \end{aligned}$$

for all sufficiently large k . Thus,

$$\text{Area of } \Omega_k > \frac{\pi}{128 K_9} \log \lambda_k,$$

for all sufficiently large k and we take $C_2 = \pi/(128 K_9)$. This proves Lemma 3.6.

3.4 The monotone slit mapping

We are now in a position to prove Lemma 3.1 and hence Theorem 1 by constructing a mapping $\mathcal{G}(z)$ from Δ to a domain \mathcal{H} , where \mathcal{H} is a modification of the domain \mathcal{D} defined by (3.3.1), (3.3.2) and Definition 3.1, so that the mapping $\mathcal{G}(z)$ satisfies (3.1.1) and (3.1.2). We begin with some notation. Let a be real and δ be positive. We define $I_{a,\delta}$ to be the interval $(a - \delta, a + \delta)$. We define $\gamma_{a,\delta}$ to be the function on $I_{a,\delta}$ whose graph is the polygonal path, having successive vertices

$$a - \delta, a - \delta/2 + i\pi/2, a + \delta/2 - i\pi/2, a + \delta.$$

Suppose that $\{\alpha_i\}_1^\infty$ and $\{\beta_i\}_1^\infty$ are two sequences of positive integers, for which $\alpha_i < \alpha_{i+1}$, $i = 1, 2, \dots$ and β_i is 'large' compared with α_i . Define a sequence of domains $\{\mathcal{D}_N\}$ corresponding to $\{\alpha_i\}$ and $\{\beta_i\}$ as follows: for each n from 1 to N and, if $n = 1$, for those a_i with $1 \leq i \leq 2^{\alpha_1}$ and, if $1 < n \leq N$, for those a_i with $2^{\alpha_{n-1}} < i \leq 2^{\alpha_n}$, define

$$I_{a_i} = I_{a_i, 2^{-\beta_n}} \text{ and } \gamma_{a_i} = \gamma_{a_i, 2^{-\beta_n}}.$$

Definition 3.2 Define \mathcal{D}_N to be the union of the sets U_1 and U_2 where

$$U_1 = \{z : \Re z \text{ is in some } I_{a_i}, i \leq 2^{\alpha_N}, \text{ and } \gamma_{a_i} - \pi < \Im z < \gamma_{a_i} + \pi\},$$

$$U_2 = \{z : \Re z \text{ is not in any } I_{a_i}, i \leq 2^{\alpha_N}, \text{ and } z \text{ is in } \mathcal{D}\}.$$

Lemma 3.1 follows directly from the lemma which follows.

Lemma 3.7 The numbers λ_i and β_i may be chosen so that for the corresponding sequence of domains $\{\mathcal{D}_N\}_0^\infty$, where $\mathcal{D}_0 = \mathcal{D}$, we have that each of the sets Ω_k of Lemma 3.6 lies in \mathcal{D}_N and that, if z is in some Ω_k , then

$$|d(0, z; \mathcal{D}_N) - d(0, z; \mathcal{D}_{N+1})| < 2^{-N}. \quad (3.4.1)$$

The domains \mathcal{D}_N clearly converge to a domain \mathcal{H} in the sense of kernel convergence and \mathcal{H} is bounded by two curves $\Gamma(t)$ and $\Gamma(t) + 2\pi i, t$ in $(-\infty, \infty)$ where $\Re \Gamma(t)$ is a strictly increasing function of t . Hence, if $\mathcal{G}(z) : \Delta \rightarrow \mathcal{H}$, where $\mathcal{G}(0) = 0$ and $\mathcal{G}'(0)$ is positive, then

$$\exp(\mathcal{G}(z)) \text{ is in } \mathcal{M}$$

which is (3.1.1). Also, each of the sets Ω_k lies in \mathcal{H} and for z in Ω_k , by Lemmas 3.6 and 3.7,

$$\lambda_k - C_1 - \sum_1^\infty \frac{1}{2^N} < d(0, z; \mathcal{H}) < \lambda_k + C_1 + \sum_1^\infty \frac{1}{2^N}.$$

Thus, for z in Ω_k ,

$$\lambda_k - C_3 < d(0, z; \mathcal{H}) < \lambda_k + C_3,$$

and, as before, the area of Ω_k exceeds $C_2 \log \lambda_k$.

We define r_k and $r'_k, k = 1, 2, \dots$ by

$$\frac{1}{2} \log \frac{1+r_k}{1-r_k} = \lambda_k - C_3 \text{ and } \frac{1}{2} \log \frac{1+r'_k}{1-r'_k} = \lambda_k + C_3.$$

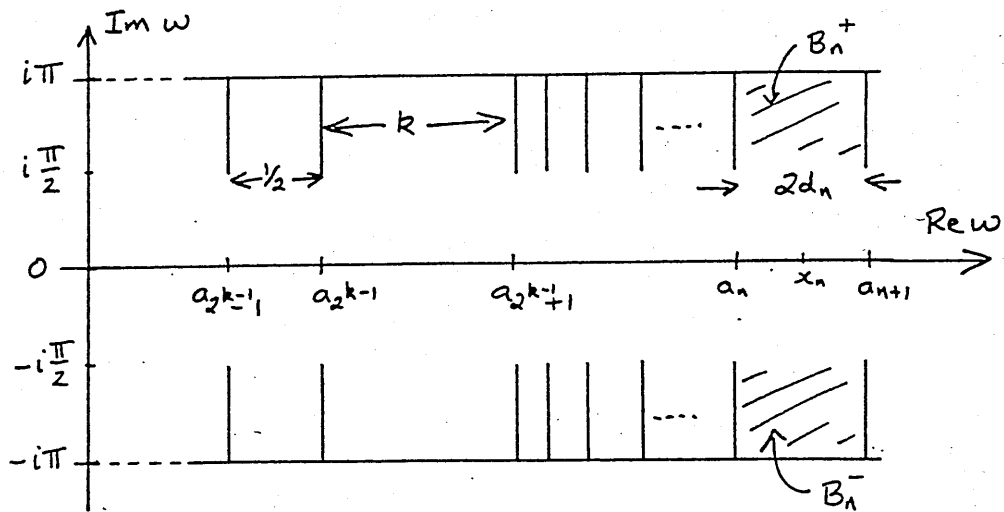


Figure 3.1 The Intermediate Domain

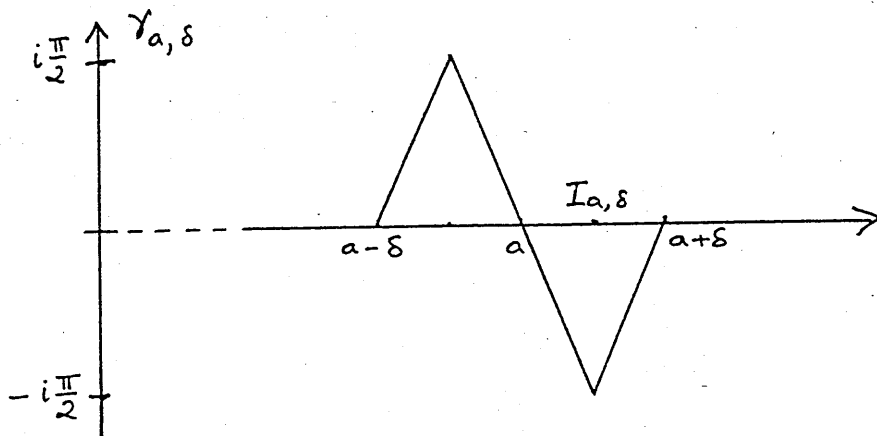


Figure 3.2 $I_{a,\delta}$ and $\gamma_{a,\delta}$

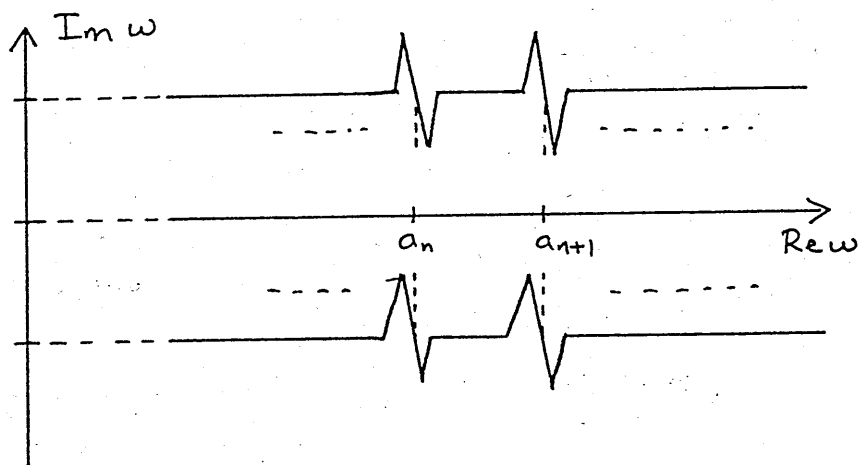


Figure 3.3 The Domain \mathcal{H}

Hence, $(1 - r_k)/(1 - r'_k)$ is bounded and

$$\begin{aligned} A(r_k, r'_k) &> \text{Area of } \Omega_k \\ &\geq C_0 \log \log \frac{1}{1 - r'_k} \end{aligned}$$

which is (3.1.2) and completes the proof of Lemma 3.1 and hence of our theorem. Thus it remains only to prove Lemma 3.7. In order to do so, we need one further result.

Lemma 3.8 *Let R_0 be the rectangle*

$$R_0 = \{s = \sigma + i\tau : -L < \sigma < L, |\tau| < \pi\}.$$

Suppose that $u_1(s), u_2(s)$ are two positive harmonic functions in R_0 , continuous on ∂R_0 and vanishing on $|\tau| = \pi$ with

$$\frac{u_1(0)}{u_2(0)} < C.$$

Then, if $-\pi < \tau < \pi$ and $L > 2 \log(24/\pi)$, we have

$$\frac{u_1(i\tau)}{u_2(i\tau)} < C(1 + 16e^{-L/2}).$$

Proof Let V be the vertical side of R_0 for which $\sigma = -L$. Then $\omega(0, V; R_0)$ is the harmonic measure of V at the origin.

Write $S_{\frac{1}{2}}$ for the half-strip $\{s = \sigma + i\tau : \sigma > -L, |\tau| < \pi\}$. It follows from the reflection principle that

$$\begin{aligned} \omega(0, V; R_0) &< \omega(0, V; S_{\frac{1}{2}}) \\ &= 2\omega(0, H; S) \end{aligned}$$

where S is the strip $\{\sigma + i\tau : |\tau| < \pi\}$ and $H = \{\sigma \pm i\pi : \sigma < -L\}$ is the boundary of S to the left of V .

The Poisson Integral formula for S gives

$$\begin{aligned} 2\omega(0, H; S) &= \frac{4e^{L/2}}{\pi} \int_{-\infty}^0 \frac{e^{\xi}}{e^{2\xi} + e^L} d\xi \\ &< \frac{4e^{L/2}}{\pi} e^{-L} \left[\int_{-\infty}^0 e^{\xi} d\xi \right] \\ &= \frac{4}{\pi} e^{-L/2}. \end{aligned}$$

When R_0 is mapped to Δ symmetrically and fixing the origin $u_1(s), u_2(s)$ in R_0 give rise to harmonic functions $u_1(z), u_2(z)$ in Δ . Moreover, by invariance of harmonic measure, $u_1(e^{it})$ and $u_2(e^{it})$ vanish when t lies in $(-\delta, \delta)$ or $(\pi - \delta, \pi + \delta)$ where $\delta = 4e^{-L/2}$.

Hence, for $0 < r < 1$,

$$\begin{aligned} u_1(\pm ir) &= \frac{1}{2\pi} \int_0^{2\pi} u_1(e^{it}) \mathcal{P}\left(r, t \mp \frac{\pi}{2}\right) dt \\ &< \frac{1}{2\pi} \left(\int_0^{2\pi} u_1(e^{it}) dt \right) \frac{1 - r^2}{1 - 2r \sin \delta + r^2} \\ &= u_1(0) \frac{1 - r^2}{1 - 2r \sin \delta + r^2}, \end{aligned}$$

where $\mathcal{P}(r, \theta)$ is the Poisson kernel in the unit disc. Similarly,

$$\begin{aligned} u_2(\pm ir) &= \frac{1}{2\pi} \int_0^{2\pi} u_2(e^{it}) \mathcal{P}\left(r, t \mp \frac{\pi}{2}\right) dt \\ &> u_2(0) \frac{1 - r^2}{1 + 2r \sin \delta + r^2}. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{u_1(\pm ir)}{u_2(\pm ir)} &< \frac{u_1(0)}{u_2(0)} \left(\frac{1 + 2r \sin \delta + r^2}{1 - 2r \sin \delta + r^2} \right) \\ &< C \frac{1 + \left(\frac{2r}{1+r^2}\right) \sin \delta}{1 - \left(\frac{2r}{1+r^2}\right) \sin \delta} \\ &< C \left(\frac{1 + \sin \delta}{1 - \sin \delta} \right), \end{aligned}$$

since $(1+x)/(1-x)$ increases on $(0, 1)$. Also,

$$\frac{1+x}{1-x} < 1 + 4x \text{ for } x < \frac{1}{2}.$$

Thus, for $\delta < \pi/6$, it follows that if $-1 < r < 1$,

$$\begin{aligned} \frac{u_1(ir)}{u_2(ir)} &< C(1 + 4 \sin \delta) \\ &< C(1 + 4\delta). \end{aligned}$$

So under the correspondence between R_0 and Δ ,

$$\frac{u_1(ir)}{u_2(ir)} < C(1 + 16e^{-\frac{1}{2}L})$$

for $-\pi < r < \pi$ if $e^{\frac{1}{2}L} > 24/\pi$. This proves Lemma 3.8.

Proof of Lemma 3.7 The sequences λ_i and β_i , whose existence is asserted in the statement of the lemma, are chosen inductively: λ_{N+1} and β_{N+1} are chosen once the domain \mathcal{D}_N has been fixed. \mathcal{D}_0 is the intermediate domain \mathcal{D} of Section 3.3; this domain is used to start the induction.

Suppose, therefore, that λ_i and $\beta_i, i = 1, 2, \dots, N$ have been chosen and let \mathcal{D}_N be the corresponding domain according to Definition 3.2. What follows works equally well when $N = 0$, $\mathcal{D}_N = \mathcal{D}$ and we wish to choose λ_1 and β_1 . To begin with, we recall from (1.2.7) that

$$d(0, z; D) = \frac{1}{2} \log \frac{1 + e^{-g(0, z; D)}}{1 - e^{-g(0, z; D)}}. \quad (3.4.2)$$

As h decreases to zero, $h(1 + e^{-h})/(1 - e^{-h})$ decreases to 2. Therefore, if ϵ is positive, there exists a positive δ , such that for $0 < h < \delta$,

$$2 < h \left(\frac{1 + e^{-h}}{1 - e^{-h}} \right) < 2 + 2\epsilon,$$

that is, for $0 < h < \delta$,

$$\log \frac{2}{h} < \log \left(\frac{1 + e^{-h}}{1 - e^{-h}} \right) < \log \frac{2}{h} + \log(1 + \epsilon). \quad (3.4.3)$$

We choose ϵ so that $\log(1 + \epsilon) = 2^{-N}$ and choose a positive δ so that (3.4.3) holds.

We recall that, for each k , the k^{th} -block of boxes is separated from the $(k + 1)^{\text{st}}$ -block by a rectangle of width 2π and length $k + 1$. Set

$$\nu_k = \frac{1}{2} (a_{2k} + a_{2k+1})$$

so that certainly ν_k increases to infinity with k . Thus, since $g(0, z; \mathcal{D}_N)$ approaches zero as $|z|$ tends to infinity, we may choose λ_{N+1} greater than λ_N so that, if $\Re z > \nu_{\lambda_{N+1}}$,

$$g(0, z; \mathcal{D}_N) < e^{-2\delta} \quad (3.4.4)$$

and so that

$$\log(1 + 16e^{-\frac{\lambda_{N+1}}{4}}) < 2^{-N-1}. \quad (3.4.5)$$

Set $V_N = \{z \text{ in } \mathcal{D}_N : \Re z > \nu_{\lambda_{N+1}}\}$. It follows from (3.4.2) and (3.4.3), that for z in V_N ,

$$\frac{1}{2} \log \frac{2}{g(0, z; \mathcal{D}_N)} < d(0, z; \mathcal{D}_N) < \frac{1}{2} \log \frac{2}{g(0, z; \mathcal{D}_N)} + \frac{1}{2^{N+1}}. \quad (3.4.6)$$

λ_{N+1} has been chosen so that $\frac{1}{2} \log(2/g)$ gives a good approximation to hyperbolic distance in V_N . It remains to choose β_{N+1} .

Let \mathcal{D}^n be the domain which corresponds to the choice, in Definition 3.2, of $\lambda_1, \lambda_2, \dots, \lambda_{N+1}$ for the λ 's and of $\beta_1, \beta_2, \dots, \beta_N, n$ for the β 's, for a positive integer n . In other words, \mathcal{D}^n is \mathcal{D}_{N+1} in the notation of Definition 3.2, with the above choice of the first $N+1$ λ 's and β 's. By allowing n to vary over the positive integers, we produce a sequence of domains $\{\mathcal{D}^n\}$. Our objective is to show that if $n = n_0$ is chosen large enough then (3.4.1) holds, in which case we put $\beta_{N+1} = n_0$. The sequence of domains $\{\mathcal{D}^n\}$ converges to \mathcal{D}_N in the sense of kernel convergence. Thus, by Theorem 1.C, if $g_n(z)$ maps \mathcal{D}^n onto Δ with $g_n(0) = 0, g'_n(0)$ positive and $g(z)$ maps \mathcal{D}_N to Δ normalised in the same way, the functions $g_n(z)$ converge to $g(z)$ uniformly on compact subsets of \mathcal{D}_N .

It follows that, for large n ,

$$\bigcup_{k=1}^{\infty} \Omega_k \subset \mathcal{D}^n$$

and that, for z in $\Omega_k, k = 1, 2, \dots, \lambda_{N+1}$,

$$|d(0, z; \mathcal{D}_N) - d(0, z; \mathcal{D}^n)| < \frac{1}{2^N}. \quad (3.4.7)$$

It remains to show that by choosing n large enough, (3.4.7) can also be made to hold for z in Ω_k when $k > \lambda_{N+1}$.

Choose a positive ϵ so that $\log(1 - \epsilon) > -2^{-N-1}$. Again by kernel convergence of $g_n(z)$ to $g(z)$, it follows that for large n ,

$$1 - \epsilon < \frac{g(0, \nu_{\lambda_{N+1}}; \mathcal{D}^n)}{g(0, \nu_{\lambda_{N+1}}; \mathcal{D}_N)} < 1 + \epsilon.$$

Since the rectangle

$$R_0 = \{z : \nu_{\lambda_{N+1}} - \lambda_{N+1}/2 < \Re z < \nu_{\lambda_{N+1}} + \lambda_{N+1}/2 \text{ and } |\Im z| < \pi\}$$

is contained in \mathcal{D}_N and since both Green's functions are harmonic there and vanish on the sides of R_0 where $|\Im z| = \pi$, it follows from Lemma 3.8 that

$$\alpha_1 = \frac{1 - \epsilon}{1 + 16e^{-\lambda_{N+1}/4}} < \frac{g(0, z; \mathcal{D}^n)}{g(0, z; \mathcal{D}_N)} < (1 + \epsilon)(1 + 16e^{-\lambda_{N+1}/4}) = \alpha_2$$

for $\Re z = \nu_{\lambda_{N+1}}$ and $|\Im z| < \pi$. Because of (3.4.5) and the choice of ϵ both $\log(1/\alpha_1)$ and $\log \alpha_2$ are less than 2^{-N} .

Both of the Green's functions $g(0, z; \mathcal{D}^n)$ and $g(0, z; \mathcal{D}_N)$ are defined in V_N and vanish on the boundary of V_N where $\Re z > \nu_{\lambda_{N+1}}$ strictly. On the remaining boundary, that is $\{z : \Re z = \nu_{\lambda_{N+1}}, |\Im z| < \pi\}$, we have

$$\alpha_1 g(0, z; \mathcal{D}_N) \leq g(0, z; \mathcal{D}^n) \leq \alpha_2 g(0, z; \mathcal{D}_N) \quad (3.4.8)$$

Hence (3.4.8) holds on the boundary of V_N and by the maximum principle throughout V_N . Since we certainly have $\log \alpha_2 < 2$ and $g(0, z; \mathcal{D}_N) < e^{-2}\delta$ in V_N by construction, $g(0, z; \mathcal{D}^n) < \delta$ throughout V_N . Therefore (3.4.6) holds for the Green's function for \mathcal{D}^n as well as for \mathcal{D}_N , that is

$$\frac{1}{2} \log \frac{2}{g(0, z; \mathcal{D}^n)} < d(0, z; \mathcal{D}^n) < \frac{1}{2} \log \frac{2}{g(0, z; \mathcal{D}^n)} + \frac{1}{2^{N+1}}. \quad (3.4.9)$$

From the inequalities (3.4.6), (3.4.8) and (3.4.9), we obtain that for z in V_N (in particular for z in $\Omega_k, k > \lambda_{N+1}$),

$$\begin{aligned} d(0, z; \mathcal{D}^n) &< \frac{1}{2} \log \frac{2}{g(0, z; \mathcal{D}^n)} + \frac{1}{2^{N+1}} \\ &< \frac{1}{2} \log \frac{2}{g(0, z; \mathcal{D}_N)} + \frac{1}{2} \log \frac{1}{\alpha_1} + \frac{1}{2^{N+1}} \\ &< d(0, z; \mathcal{D}_N) + \frac{1}{2^N}, \\ d(0, z; \mathcal{D}^n) &> \frac{1}{2} \log \frac{2}{g(0, z; \mathcal{D}^n)} \\ &> \frac{1}{2} \log \frac{2}{g(0, z; \mathcal{D}_N)} - \frac{1}{2} \log \alpha_2 \\ &> d(0, z; \mathcal{D}_N) - \frac{1}{2^N}. \end{aligned}$$

Hence for sufficiently large n , say $n = n_0$, and z in $\Omega_k, k = 1, 2, \dots$,

$$|d(0, z; \mathcal{D}_N) - d(0, z; \mathcal{D}^n)| < \frac{1}{2^N}.$$

Choose $\beta_{N+1} = n_0$. This completes the proof of Lemma 3.7.

Chapter 4

Background on the angular derivative problem

4.1 Definitions and a basic theorem

If $f(z)$ is analytic in a neighbourhood of the point z_0 then, by the definition of the derivative,

$$|f(z) - f(z_0) - (z - z_0)f'(z_0)| = o(|z - z_0|) \quad (4.1.1)$$

as $z \rightarrow z_0$. Suppose now that $f'(z_0)$ is non-zero. Then it follows that $f(z)$ is conformal at z_0 , i.e. for z near z_0 ,

$$\arg(f(z) - f(z_0)) = \arg(z - z_0) + \arg(f'(z_0)) + o(1). \quad (4.1.2)$$

It also follows that

$$\lim_{z \rightarrow z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = |f'(z_0)|. \quad (4.1.3)$$

Moreover, (4.1.2) and (4.1.3) with $f'(z_0)$ non-zero imply (4.1.1).

The situation where z_0 is a boundary point of the domain of analyticity of $f(z)$ has received much attention. We consider a conformal mapping $f(z)$ from a fixed domain, such as the unit disc Δ , onto a general simply connected domain D and we wish to find geometric conditions on the boundary of D near $f(1)$ so that (4.1.2) and (4.1.3) hold with $z_0 = 1$. This is the context of the angular derivative problem.

We take our fixed domain to be the upper half-plane H and the boundary point of H we consider is the origin. Now we define the angular limit at 0 of a function $g(z)$. If $0 < \alpha < \pi$, we write

$$A(\alpha) = \left\{ z : \frac{\pi}{2} - \frac{\alpha}{2} < \arg z < \frac{\pi}{2} + \frac{\alpha}{2} \right\}.$$

We say that $g(z)$ has the angular limit $g(0)$ if

$$\lim_{z \rightarrow 0, z \in A(\alpha)} g(z) = g(0)$$

for each angle $A(\alpha)$ with $0 < \alpha < \pi$. Suppose that $f(z)$ is a conformal mapping from H to a domain D . We say that $f(z)$ has the angular derivative $f'(0)$ at 0 if $f'(z)$ has the angular limit, $f'(0)$, at 0.

Note that the existence of an angular derivative is a local property of the domain at a point rather than of the particular conformal mapping $f(z)$.

The angular derivative problem is to find geometric conditions on the domain D so that $f : H \mapsto D$ has a non-zero angular derivative at 0.

The existence of an angular derivative implies the existence of an angular limit. Suppose, in fact, that the angular derivative of f exists. First, the radial limit

$$f(0) = \lim_{y \rightarrow 0} f(iy)$$

exists since

$$f(iy) = f(i) - i \int_y^1 f'(i\rho) d\rho.$$

Next, suppose that z is in $A(\alpha)$ for some α in $(0, \pi)$. Then, if $z = x + iy$,

$$\begin{aligned} |f(z) - f(0)| &\leq |f(z) - f(iy)| + |f(iy) - f(0)| \\ &= \left| \int_0^x f'(\rho + iy) d\rho \right| + |f(iy) - f(0)| \\ &\leq x(|f'(0)| + o(1)) + |f(iy) - f(0)| \\ &< Cy(|f'(0)| + o(1)) + |f(iy) - f(0)| \end{aligned}$$

which has limit 0 as y tends to zero or equivalently as z approaches 0 in $A(\alpha)$.

We say that $f(z)$ has the asymptotic value a at the boundary point ζ of H if there is a Jordan arc Γ that ends at ζ and otherwise lies in H , for which $f(z)$ approaches a when z approaches ζ along Γ .

We now quote the very useful Theorem 10.5 of [28].

Theorem 4.A *Let $f(z)$ be a conformal mapping of the upper half-plane H onto D for which $f(z)$ has the angular limit $f(0)$ at 0. Then the following propositions are equivalent.*

1. $f(z)$ has the angular derivative $f'(0)$ at 0;
2. $f'(z)$ has the asymptotic value $f'(0)$ at 0;
3. $(f(z) - f(0))/z$ has the angular limit $f'(0)$ at 0;
4. $(f(z) - f(0))/z$ has the asymptotic value $f'(0)$ at 0.

Theorem 4.A shows that the existence of a non-zero angular derivative implies conformality at the boundary. So, corresponding to (4.1.2), we see that if f has a non-zero angular derivative at 0 then Theorem 4.A (3) implies that

$$\arg(f(z) - f(0)) = \arg z + \arg f'(0) + o(1) \quad (4.1.4)$$

when z is in $A(\alpha)$ for any fixed α in $(0, \pi)$. More easily, the angular limit

$$\lim_{z \rightarrow 0} \frac{|f(z) - f(0)|}{|z|}$$

exists and equals $|f'(0)|$.

4.2 Partial results

Partial results have been obtained by many authors and we describe some of these results now. It is practically impossible to mention each of the results on the angular derivative problem as the literature is extensive and many earlier results have been superseded.

Ostrowski [27] has given necessary and sufficient conditions for conformality, that is for (4.1.4) to hold.

Jenkins and Oikawa [24] and Rodin and Warschawski [31] have given a necessary and sufficient condition for the existence of an angular derivative in terms of extremal length. Then, in [30], Rodin and Warschawski use extremal length estimates and the extremal length criterion to extend and to give simpler proofs of earlier results. In particular, they give complete solutions to the angular derivative problem when D contains or is contained in a half-plane. Their results generalise theorems of Ferrand in [17] and [25]. For example, Condition 2 of Theorem 4.C is due to her. It is worth stating these theorems explicitly here if only to give a flavour for the type of result known. We need some notation to do so.

Let D denote a simply connected domain which contains the line segment $[0, r_0]$ for some positive r_0 . For small r , denote by θ_r the largest arc of $D \cap \{|z| = r\}$ which contains ir . We choose $\arg z$ to lie in $(-\pi/2, 3\pi/2]$.

Let u_n be a decreasing sequence of positive numbers with limit 0. Put

$$\delta_n = \log \frac{u_n}{u_{n+1}},$$

$$v'(r) = \min\{\arg z : z \text{ is in } \theta_r\},$$

$$v''(r) = \max\{\arg z : z \text{ is in } \theta_r\},$$

$$\theta'_n = \max\{|v'(r)| : u_{n+1} \leq r \leq u_n\},$$

$$\theta''_n = \max\{|v''(r) - \pi| : u_{n+1} \leq r \leq u_n\}.$$

We can now state [30], Theorem 1.

Theorem 4.B *Let D be a simply connected domain contained in the upper half-plane H and for which 0 is accessible along the positive imaginary axis. Then the following conditions are equivalent.*

1. D has a non-zero angular derivative at 0.
2. Any sequence of positive, decreasing numbers $\{u_n\}$ with $\sum \delta_n^2$ finite satisfies

$$\sum \delta_n \theta'_n < \infty \text{ and } \sum \delta_n \theta''_n < \infty.$$

3. There exists a positive, decreasing sequence $\{u_n\}$ with

$$\sum \delta_n^2 < \infty, \sum (\theta'_n)^2 < \infty \text{ and } \sum (\theta''_n)^2 < \infty.$$

4. There exists a positive, decreasing sequence $\{u_n\}$ with

$$\sum \delta_n \theta'_n < \infty, \sum \delta_n \theta''_n < \infty, \sum (\theta'_n)^2 < \infty \text{ and } \sum (\theta''_n)^2 < \infty.$$

5. There exists a covering of $(H \setminus D) \cap \{|z| < 1\}$ by discs Δ_n of radius r_n centred at x_n on the real axis such that

$$\sum \left(\frac{r_n}{x_n} \right)^2 < \infty.$$

The theorem which follows deals with the case when D contains a half-plane. It is Theorem 2 of [30].

Theorem 4.C Suppose that D is a simply connected domain which contains the upper half-plane H . Then the following conditions are equivalent.

1. D has a non-zero angular derivative at 0.
2. Whenever $D \cap \{|z| < 1\}$ contains disjoint discs Δ_n centred at x_n on the real axis and radius r_n then

$$\sum \left(\frac{r_n}{x_n} \right)^2$$

must be finite.

3. There is a positive, decreasing sequence $\{u_n\}$ such that

$$\sum \delta_n^2 < \infty, \sum (\theta'_n)^2 < \infty \text{ and } \sum (\theta''_n)^2 < \infty.$$

4. There is a positive, decreasing sequence $\{u_n\}$ such that

$$\sum \delta_n^2 < \infty, \sum \delta_n \theta'_n < \infty \text{ and } \sum \delta_n \theta''_n < \infty.$$

A theorem due to B. G. Eke gives a necessary and sufficient condition for the existence of an angular derivative for a special class of domains which, in contrast to the results above, need not contain or be contained in a half-plane. Partial results for this class had been obtained earlier by Ferrand and Dufresnoy and by Warschawski. The angular derivative problem in general remains unsolved however.

Eke considers domains which arise in the following manner. Suppose that $\{u_n\}$ is a sequence of positive numbers for which $u_{n+1} \leq \lambda u_n$ where $0 < \lambda < 1$. Suppose that $\{v_n\}$ and $\{v'_n\}$ are sequences of real numbers for which

$$\lim_{n \rightarrow \infty} v_n = 0 \text{ and } \lim_{n \rightarrow \infty} v'_n = \pi$$

and $v_n < v'_n$. Then the simply connected domain D is to be the interior of the union of the sectors

$$\{z : u_{n+1} \leq |z| \leq u_n \text{ and } v_n \leq \arg z \leq v'_n\}.$$

We say that D is in class E . He defines

$$\theta_n = v'_n - v_n,$$

$$\lambda_n = \max\{|v'_{n+1} - v'_n|, |v_{n+1} - v_n|\}.$$

In [16], Eke proves the following.

Theorem 4.D *If D is a simply connected domain in class E for which either*

$$\sum_{n=1}^{\infty} \left(\frac{\pi - \theta_{n+1}}{\theta_{n+1}} \right) (u_{n+1} - u_n) \quad (4.2.1)$$

or

$$\sum_{n=1}^{\infty} \lambda_n^2 \log \frac{1}{\lambda_n} \quad (4.2.2)$$

is convergent, then a necessary and sufficient condition for D to have a non-zero angular derivative at 0 is the convergence of the other sum.

A special case of the above result is when $v'_n = v_n + \pi$ for each n . Then D is the interior of the union of the sectors

$$\{z : u_{n+1} \leq |z| \leq u_n \text{ and } v_n \leq \arg z \leq v_n + \pi\}$$

where, as before, $u_{n+1} \leq \lambda u_n$ and $\lim v_n = 0$. In this case $\lambda_n = |v_{n+1} - v_n|$ and Theorem 4.D yields

Corollary 4.A *The above domain has a non-zero angular derivative at 0 if and only if*

$$\sum_{n=1}^{\infty} \lambda_n^2 \log \frac{1}{\lambda_n}$$

is convergent.

4.3 Burdzy's results

Burdzy approaches the angular derivative problem through probability theory and modern potential theory. His papers [4], [5] and [6] are a trilogy under the global title of 'Brownian excursions and minimal thinness'. In the first in the series he establishes results on Brownian excursion laws and derives criteria for minimal thinness from these results. The second article presents applications of these results to the boundary behaviour of the Green's function and the third presents applications to the angular derivative problem. We outline each article in turn.

In [4], the methods used to establish the criteria for minimal thinness involve Brownian excursion laws and potential theory. Brownian excursion laws, Burdzy says, form part of the exit theory of Maisonneuve which is a generalisation of the excursion theory of Markov processes. The main references for this are Maisonneuve's article in *Ann. Probab* 3. (1975) and the book 'Markov processes and Martingales, Vol. 1' by D. Williams. The main references for the potential theory used is Doob's book 'Classical Potential Theory and its Probabilistic Counterpart' and the book 'Brownian Motion and Classical Potential Theory' by Port and Stone.

The major part of [4] is taken up with results on Brownian excursions. Some results from [7] are also needed. The last section deals with criteria for minimal thinness where the following theorem, Theorem 3.2, is stated and proved using results established earlier in the paper. It is assumed that d is greater than 1.

Theorem 4.E Let $h : F \rightarrow \mathcal{R}$ be a nonnegative function and put

$$A = \{(X, y) \text{ in } H : 0 < y \leq h(X)\}.$$

Suppose that h is Lipschitz or that $h(X) = h_1(|X|)$ for some monotone function $h_1 : [0, \infty) \rightarrow \mathcal{R}$. Then A is minimal thin at 0 if and only if

$$\int_{|T| \leq 1} \frac{h(T)}{|T|^d} dT < \infty.$$

In [5], Burdzy uses the material in [4] to obtain results on the boundary behaviour of the Green's function. He proves (Theorem 4.1 and Theorem 4.2 of [5])

Theorem 4.F Suppose that either the simply connected domain D is contained in H and satisfies the cone condition

$$\{(X, y) : y > c|X| \text{ and } y < \frac{1}{c}\} \subseteq D, \quad (c > 0),$$

or that D contains the half-space H . Fix P in $D \cap H$. Then the limit

$$\lim_{y \rightarrow 0} \frac{g(P, (0, y); D)}{y}$$

exists.

In the author's opinion, the main results of [5] are Corollaries 4.1, 4.2 and 4.3 which we state now.

Suppose that

$$D = \{(X, y) : y > h(X)\}$$

for some nonnegative function $h : F \mapsto \mathcal{R}$ such that $h(0) = 0$. Fix P in D . Then Corollary 4.1 runs as follows.

Theorem 4.G If $h(X)$ is Lipschitz or $h(X) = h_1(|X|)$ for some monotone function $h_1 : [0, \infty) \mapsto \mathcal{R}$ for which $h_1(t) < ct$ if $t < 1/c$ for some positive c , then the limit

$$\lim_{y \rightarrow 0} \frac{g(P, (0, y); D)}{y}$$

is greater than 0 if and only if

$$\int_{|T|<1} \frac{h(T)}{|T|^d} dT < \infty. \quad (4.3.1)$$

The author continues with Corollary 4.2 to Theorem 4.2, the proof of which he admits is (some eleven pages) long.

Suppose that $h : F \mapsto \mathcal{R}$ is nonnegative and that $h(0) = 0$. Suppose that

$$D = \{(X, y) : y > -h(X)\}.$$

Fix P in H .

Theorem 4.H *Suppose that $h(X)$ is Lipschitz. Then*

$$\lim_{y \rightarrow 0} \frac{g(P, (0, y); D)}{y} < \infty$$

if and only if

$$\int_{|T| \leq 1} \frac{h(T)}{|T|^d} dT < \infty. \quad (4.3.2)$$

Under the same assumptions he states Corollary 4.3.

Theorem 4.I *Suppose that $h : F \mapsto \mathcal{R}$ is given by $h(X) = h_1(|X|)$ for some nonnegative monotone function $h_1 : [0, \infty) \mapsto \mathcal{R}$. Then*

$$\lim_{y \rightarrow 0} \frac{g(P, (0, y); D)}{y} < \infty$$

if and only if

$$\int_{|T| \leq 1} \frac{h(T)}{|T|^d} dT < \infty. \quad (4.3.3)$$

In [6], the author begins by reviewing the results of the earlier papers and gives some preliminary results on the angular derivative problem. The main result of this paper is Theorem 7.1.

Burdzy introduces the concept of a Lipschitz majorant to the boundary of a simply connected plane domain D . He supposes that 0 is a boundary point of D and fixes a positive number ϵ . Then h_ϵ is defined to be the smallest Lipschitz function with constant 1 for which $\partial D \cap \{z : |z| < \epsilon\}$ is contained in $\{z : \Im z \leq h_\epsilon(\Re z)\}$.

For a real-valued function g , denote the function $\max\{0, g\}$ by g^+ and denote $-\min\{0, g\}$ by g^- .

Theorem 7.1 of [6] runs as follows.

Theorem 4.J *Suppose that D is a simply connected domain which has 0 as a boundary point and that the Lipschitz majorant h_ϵ of the part of the boundary of D in $\{z : |z| < \epsilon\}$ satisfies*

$$\int_{-1}^1 \frac{h_\epsilon^+(t)}{t^2} dt < \infty \quad (4.3.4)$$

for some positive ϵ .

Then the angular derivative $f'(0)$ exists. It is not equal to 0 if and only if

$$\int_{-1}^1 \frac{h_\epsilon^-(t)}{t^2} dt < \infty. \quad (4.3.5)$$

To prove Theorem 4.J, Burdzy uses, directly or indirectly, all of the results listed above together with others from [7] and many others from probability theory. It is our aim in Chapter 5 to give an elementary proof of the necessity of condition (4.3.5).

The first to give classical proofs of some of Burdzy's results were Rodin and Warschawski. In [32], they showed the equivalence of Burdzy's Lipschitz majorant conditions to some of the older notions used as criteria for the existence of angular derivatives, which we described earlier. They proved, ([32], Theorem 1),

Theorem 4.K *Suppose that D is a simply connected domain which has 0 as a boundary point and, for a positive ϵ , let h_ϵ denote the Lipschitz majorant of the part of the boundary of D in $\{z : |z| < \epsilon\}$. Then Condition*

2 of Theorem 4.C implies (4.3.5). If $D \cap H$ has an angular derivative at 0 then (4.3.5) implies Condition 2 of Theorem 4.C.

The condition that $D \cap H$ has an angular derivative at 0 is equivalent to (4.3.4).

Thus we see by Theorem 4.B that (4.3.4) and Condition 5 of Theorem 4.B are equivalent. Using known results and techniques, they then proved

Theorem 4.L *Suppose that D is a simply connected domain which has 0 as a boundary point. Suppose that the Lipschitz majorant h_ϵ of the boundary of D near 0 satisfies*

$$\int_{-1}^1 \frac{h_\epsilon^+(t)}{t^2} dt < \infty$$

and

$$\int_{-1}^1 \frac{h_\epsilon^-(t)}{t^2} dt < \infty$$

for some positive ϵ . Then the angular derivative $f'(0)$ exists and is non-zero.

4.4 Theorem 4.1 and the connection with Burdzy's work

Let $h(X)$ be a real-valued Lip 1 function defined on F which vanishes outside $|X| < 1$ and for which $h(0) = 0$. Suppose that $h(X)$ has Lipschitz constant 1 so that $|h(X_1) - h(X_2)| \leq |X_1 - X_2|$. Then

$$\mathcal{D} = \{(X, y) \text{ in } \mathcal{R}^d : y > h(X)\} \quad (4.4.1)$$

is a domain in \mathcal{R}^d with 0 in $\partial\mathcal{D}$.

As before, write

$$h^+(X) = \max\{0, h(X)\}, \quad h^-(X) = -\min\{0, h(X)\}.$$

For positive ϵ write \mathcal{D}_ϵ for the component of $\mathcal{D} \cap B(0, \epsilon)$ which contains those points $(0, y)$ with $0 < y < \epsilon$.

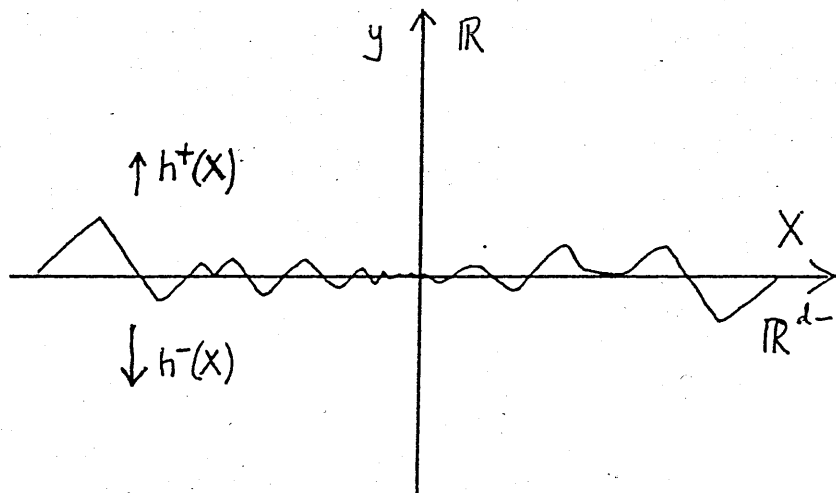


Figure 4.1 The domain D of (4.4.1)

Theorem 4.1 *Suppose that the integral*

$$\int_{|T|<1} \frac{h^+(T)}{|T|^d} dT < \infty \quad (4.4.2)$$

and that

$$\int_{|T|<1} \frac{h^-(T)}{|T|^d} dT = \infty. \quad (4.4.3)$$

If ϵ is positive and if u is any positive harmonic function in D_ϵ which vanishes at all points of $\partial D \cap \partial D_\epsilon$, then

$$\frac{u(0, y)}{y} \rightarrow \infty \text{ as } y \rightarrow 0^+. \quad (4.4.4)$$

In Chapter 5 we give an elementary, non-probabilistic proof of Theorem 4.1. We now show that this leads easily to a proof of the necessity of condition (4.3.5) in Theorem 4.J, thereby complementing Rodin and Warschawski's Theorem 4.L.

Let f and D satisfy the assumptions of Theorem 4.J and suppose that (4.3.4) holds but that (4.3.5) does not. We want to conclude that $f'(0) = 0$.

By the definition of $h_\epsilon(x)$,

$$D'_\epsilon = \{z : y > h_\epsilon(x) \text{ and } |z| < \epsilon\}$$

is contained in D . Writing D_ϵ for the component of D'_ϵ containing z_ϵ , we conclude from Theorem 4.1 that if u is positive and harmonic in D_ϵ and vanishes on $\{z \in \partial D_\epsilon : y = h_\epsilon(x)\}$ then (4.4.4) holds, i.e.

$$\frac{u(zy)}{y} \rightarrow \infty \text{ as } y \rightarrow 0^+.$$

Next we show that any conformal map f_ϵ of the upper half-plane H onto D_ϵ with $f_\epsilon(0) = 0$ has $f'_\epsilon(0) = 0$. Let g_ϵ denote the inverse function of f_ϵ . Since

$$u(z) = \Im g_\epsilon(z)$$

is positive and harmonic in D_ϵ and vanishes on $\{z \in \partial D_\epsilon : y = h_\epsilon(x)\}$,

$$\left| \frac{g_\epsilon(zy)}{zy} \right| \geq \frac{\Im g_\epsilon(zy)}{y} \rightarrow \infty \text{ as } y \rightarrow 0^+,$$

by Theorem 4.1. Thus, putting $w_y = g_\epsilon(zy)$,

$$\left| \frac{f_\epsilon(w_y)}{w_y} \right| \rightarrow 0 \text{ as } y \rightarrow 0^+,$$

that is, $f_\epsilon(w)/w$ has the asymptotic value 0 along the asymptotic path $\Gamma(t) = w_t, 0 < t < \epsilon/2$ in H . Thus, by Theorem 4.A, $f'_\epsilon(0)$ exists and $f'_\epsilon(0) = 0$. Since $D_\epsilon \subset D$, we deduce from a standard comparison theorem (see e.g. [28], Theorem 10.6) that $f'(0) = 0$ also. Thus condition (4.3.5) is necessary in Theorem 4.J.

Theorem 4.1 also gives a proof of the necessity of condition (4.3.1) and (4.3.2) in Theorems 4.G and 4.H respectively, and hence of condition (4.3.3) in Theorem 4.I. This is because the monotone function $h_1(X)$ lies between two Lipschitz functions $h_u(X)$ and $h_l(X)$ for which, according as the integral in (4.3.3) converges or diverges, the corresponding integrals converge or diverge. In fact, Theorem 4.1 generalises the necessity of condition (4.3.5) in Theorem 4.J to space. Burdzy himself does the same in a later paper with R. J. Williams [8].

Chapter 5

A non-probabilistic proof of Theorem 4.1

5.1 Introduction

We present in this chapter an elementary and non-probabilistic proof of Theorem 4.1 on the boundary behaviour of positive harmonic functions near a boundary point of a Lipschitz domain. The significance of this result was discussed in Chapter 3. We present the result again here for the reader's convenience.

It is supposed that $h(X)$ is a real-valued Lip 1 function with Lipschitz constant 1, which is defined on F , vanishes outside $|X| < 1$ and for which $h(0) = 0$. We put

$$D = \{(X, y) \in \mathcal{R}^d : y > h(X)\} \quad (5.1.1)$$

For positive ϵ we write D_ϵ for the component of $D \cap B(0, \epsilon)$ which contains those points $(0, y)$ with $0 < y < \epsilon$.

Theorem 4.1 *Suppose that the integral*

$$\int_{|T|<1} \frac{h^+(T)}{|T|^d} dT < \infty \quad (5.1.2)$$

and that

$$\int_{|T|<1} \frac{h^-(T)}{|T|^d} dT = \infty. \quad (5.1.3)$$

If ϵ is positive and if u is any positive harmonic function in D_ϵ which

vanishes at all points of $\partial D \cap \partial D_\epsilon$, then

$$\frac{u(0, y)}{y} \rightarrow \infty \text{ as } y \rightarrow 0^+. \quad (5.1.4)$$

The remainder of this chapter is devoted to the proof of this theorem.

5.2 Some technical lemmas

We begin with a simple estimate for the Poisson kernel. Let $P(X, y)$ denote the Poisson kernel of the ball $B(0, r)$ with its singularity at $(0, r)$ and normalised so that $P(0) = 1$. In other words, for $(X, y) \in B(0, r)$,

$$P(X, y) = r^{d-2} \frac{r^2 - (|X|^2 + y^2)}{[|X|^2 + (r - y)^2]^{\frac{d}{2}}}. \quad (5.2.1)$$

Lemma 5.1 *For $|X| < r$, we have*

$$\frac{\partial P}{\partial y} \Big|_{(X, 0)} \leq \frac{d}{r}. \quad (5.2.2)$$

Proof A routine calculation gives

$$\begin{aligned} \frac{\partial P}{\partial y} \Big|_{(X, 0)} &= r^{d-2} \left\{ \frac{d(r^2 - |X|^2)r}{(|X|^2 + r^2)^{\frac{d+2}{2}}} \right\} \\ &\leq r^{d-2} \left\{ \frac{dr^2 r}{r^{d+2}} \right\} \\ &= \frac{d}{r}. \end{aligned}$$

This proves Lemma 5.1.

The next lemma gives a criterion for a function $u(X, y)$, which is harmonic on both sides of a hyperplane, to be subharmonic across the hyperplane. Such a lemma was, for example, given by D. Drasin [12] in the case $d = 2$. We include a proof for completeness. Define

$$B^+ = \{(X, y) \in B(0, 1) : y > 0\},$$

$$B^- = \{(X, y) \in B(0, 1) : y < 0\}.$$

Lemma 5.2 Suppose that $u(X, y)$ is real-valued and continuous in the ball $B(0, 1)$ and is harmonic in both B^+ and B^- . If for each $(X, 0)$ in $B(0, 1)$,

$$D_+u(X) = \lim_{y \rightarrow 0^+} \frac{u(X, y) - u(X, 0)}{y} \quad (5.2.3)$$

and

$$D_-u(X) = \lim_{y \rightarrow 0^+} \frac{u(X, 0) - u(X, -y)}{y} \quad (5.2.4)$$

exist, and satisfy

$$D_+u(X) \geq D_-u(X), \quad (5.2.5)$$

then $u(X, y)$ is subharmonic in $B(0, 1)$.

Proof We will show that for each $(X, 0)$ with $|X| < 1$ and all sufficiently small r , $u(X, 0)$ is dominated by the average of its boundary values on $\partial B((X, 0), r)$. Then, by definition, (see e.g. [22] page 40), $u(X, y)$ will be subharmonic at each such point $(X, 0)$ and since, by assumption, $u(X, y)$ is harmonic in B^+ and B^- , it will then follow that $u(X, y)$ is subharmonic throughout $B(0, 1)$. Note that u is continuous on the closure of $B((X, 0), r)$ by assumption. For simplicity we take $X = 0$ in the following argument.

Suppose first that strict inequality holds in (5.2.5). Let h be harmonic in $B(0, r)$, continuous in $\overline{B(0, r)}$ and equal to u on $\partial B(0, r)$. We show that $u \leq h$ in $B(0, r)$. Suppose contrary to this that $u > h$ somewhere in $B(0, r)$. Then $u - h$ attains its positive maximum m at $(X_0, y_0) \in B(0, r)$. Thus (X_0, y_0) cannot be in $\partial B(0, r)$. If $y_0 > 0$, then $u - h$ has the constant value m in $B(0, r) \cap \{y > 0\}$, contrary to $u - h = 0$ on part of the boundary. Thus $y_0 \leq 0$. Similarly $y_0 \geq 0$, so that $y_0 = 0$.

Write $v(X, y) = u(X, y) - h(X, y)$, $\phi(y) = v(X_0, y)$. Then

$$\phi(y) \leq \phi(0) = m \text{ and } \phi(-y) \leq \phi(0) = m.$$

Thus,

$$0 \geq \lim_{y \rightarrow 0} \frac{\phi(y) + \phi(-y) - 2\phi(0)}{y} = \lim_{y \rightarrow 0} \frac{u(X_0, y) + u(X_0, -y) - 2u(X_0, 0)}{y}.$$

This contradicts (5.2.5) with strict inequality.

If only weak inequality holds in (5.2.5) we apply the above argument to $u + \epsilon|y|$, and deduce that $u \leq h + \epsilon r$. Letting ϵ tend to zero, we obtain $u \leq h$, so that this holds in all cases. Now putting $(X, y) = 0$ we obtain the mean-value inequality so that u is subharmonic.

The last lemma in this section shows that in order to prove Theorem 4.1 it is enough to construct a single function $U(X, y)$ in D which is nonpositive on ∂D , subharmonic near 0 and which satisfies (5.1.4).

Suppose then that D is given by (5.1.1) where $h(X)$ satisfies (5.1.2) and (5.1.3).

Lemma 5.3 *Suppose that there is a function $U(X, y)$ in D which is non-positive on the boundary of D , is subharmonic in $D \cap B(0, \epsilon_0)$, where $\epsilon_0 > 0$, and for which*

$$\frac{U(0, y)}{y} \rightarrow \infty \text{ as } y \rightarrow 0^+, \quad (5.2.6)$$

then Theorem 4.1 follows.

The following boundary Harnack principle will be needed. It was proved by B. Dahlberg [11] and J.-M. Wu [35] independently. Ancona [1] extended the principle to certain elliptic operators. In the plane case the result is easily obtained by using conformal mapping. When $d > 2$, however, the full strength of the proofs in [11] and [35] is needed. A domain D in \mathcal{R}^d is said to be a Lipschitz domain if D is a bounded domain and to each $(X_0, y_0) \in \partial D$ there corresponds a local coordinate system (X, y) ($X \in \mathcal{R}^{d-1}$ and $y \in \mathcal{R}$) and a Lipschitz function f from \mathcal{R}^{d-1} to \mathcal{R} such that

$$N \cap D = \{(X, y) : y > f(X)\} \cap D$$

for some neighbourhood N of (X_0, y_0) . The boundary Harnack principle is as follows.

Theorem 5.A (Dahlberg, Wu) *Suppose that D is a Lipschitz domain, that (X_0, y_0) is a point in D , E a relatively open set on ∂D and S is a sub-domain of D satisfying $\partial S \cap \partial D \subseteq E$. Then there is a constant C , so that*

whenever u_1 and u_2 are two positive harmonic functions in D vanishing on E and $u_1(X_0, y_0) = u_2(X_0, y_0)$, then

$$u_1(X, y) \leq C u_2(X, y)$$

for all $(X, y) \in S$.

Proof of Lemma 5.3 Since the components D_ϵ are nested, i.e. $D_{\epsilon_1} \subseteq D_{\epsilon_2}$ for $\epsilon_1 < \epsilon_2$, we may assume that $\epsilon < \epsilon_0$.

Suppose, then, that u is a positive harmonic function in D_ϵ which vanishes at all points of $\partial D \cap \partial D_\epsilon$. Note that U restricted to D_ϵ is subharmonic and nonpositive on $\partial D \cap \partial D_\epsilon$. Put

$$K = \max\{U(X, y) : |(X, y)| = \epsilon \text{ and } (X, y) \in \partial D_\epsilon\}.$$

Then, by the Principle of harmonic measure ([22], Theorem 3.11),

$$U(X, y) \leq K \omega(X, y)$$

where $\omega(X, y)$ is the harmonic measure of $\partial D_\epsilon \cap \{|(X, y)| = \epsilon\}$ with respect to D_ϵ . By (5.2.6), therefore,

$$\frac{\omega(0, y)}{y} \rightarrow \infty \text{ as } y \rightarrow 0^+.$$

Since by Theorem 5.A, $u(0, y) \geq C \omega(0, y)$ for all small y , (5.1.4) follows. This completes the proof of Lemma 5.3.

5.3 A geometric lemma

We begin this section with a lemma which will be needed in the proof of Lemma 5.6 and which, in the plane, is equivalent to standard results on the angular derivative problem, (see [32], Theorem C and Theorem 1(iv)). Let $h(T)$ be a nonnegative Lipschitz function on F with Lipschitz constant $k = 1$ such that $h(0) = 0$. Let D be the domain

$$D = \{(X, y) : y > h(X)\}. \quad (5.3.1)$$

Lemma 5.4 *Suppose that*

$$\int_{|T|<1} \frac{h(T)}{|T|^d} dT = \infty. \quad (5.3.2)$$

If $u(X, y)$ is continuous in \overline{D} , positive and harmonic in D and vanishes on ∂D in a neighbourhood of 0 then

$$\frac{u(0, y)}{y} \rightarrow 0 \text{ as } y \rightarrow 0^+. \quad (5.3.3)$$

Lemma 5.4 is proved using the theory of Sections 1.2.7 and 1.2.8 and following closely an argument used by Rippon [29] which is in turn based on work of Warschawski [34]. To do this, we first of all need the lemma which is to follow.

Lemma 5.5 *Let $S(E_r)$ denote the area of the spherical cap*

$$E_r = \{(X, y) \in D : |X|^2 + y^2 = r^2\}.$$

Then

$$S(E_r) \leq \frac{c_d}{2} r^{d-1} \left(1 - k \int_{|T|=r} \frac{h(T)}{|T|^{d-1}} d\sigma \right)$$

for all small r , where $d\sigma$ denotes $(d-1)$ -dimensional measure on $|T|=r$ and k is a constant depending only on d .

Proof The cap E_r is connected since $h(T)$ has Lipschitz constant 1 and so E_r meets each hyperplane containing the y -axis in an arc. Suppose that $h(T_0) \neq 0$, where $T_0 \in F$. Since $h(T)$ has Lipschitz constant 1,

$$h(T) > \frac{1}{2} h(T_0) \text{ for } |T - T_0| < \frac{1}{2} h(T_0).$$

Thus $(T_0 \sin \theta, |T_0| \cos \theta) \notin D$ for $\frac{\pi}{2} \geq \theta \geq \theta_{T_0}$ where θ_{T_0} is defined by

$$|T_0| \cos \theta_{T_0} = \frac{1}{2} h(T_0).$$

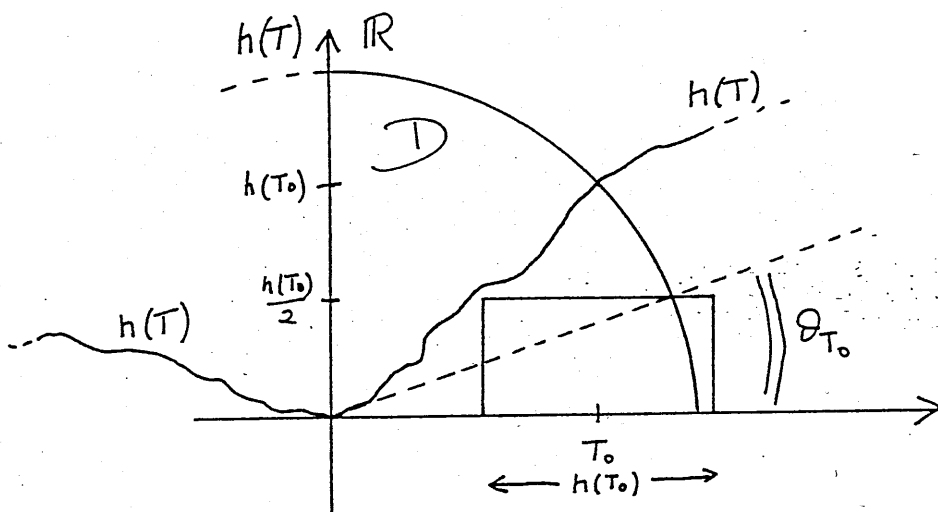


Figure 5.1 Diagram for Lemma 5.5

The element of surface measure at the point $(T \sin \theta, |T| \cos \theta)$ on the surface of the d -sphere of radius r is

$$(rd\theta)(\sin \theta)^{d-2} d\sigma$$

where $d\sigma$ denotes $(d-2)$ -dimensional measure on $|T| = r$.

It follows that

$$S(E_r) \leq \frac{c_d}{2} r^{d-1} - \int_{|T|=r} \int_{\theta_T}^{\frac{\pi}{2}} r (\sin \theta)^{d-2} d\theta d\sigma$$

where $d\sigma$ denotes $(d-1)$ -dimensional measure on $|T| = r$.

Since $\sin \theta \geq \frac{2}{\pi} \theta$,

$$\begin{aligned} \int_{\theta_T}^{\frac{\pi}{2}} (\sin \theta)^{d-2} d\theta &\geq \left(\frac{2}{\pi}\right)^{d-2} \frac{1}{d-1} \left\{ \left(\frac{\pi}{2}\right)^{d-1} - \theta_T^{d-1} \right\} \\ &\geq k \left(\frac{\pi}{2} - \theta_T\right), \end{aligned}$$

for a constant k depending only on d . Note that

$$\frac{\pi}{2} - \theta_T \geq \frac{1}{2} \frac{h(T)}{r},$$

and so

$$\begin{aligned} S(E_r) &\leq \frac{c_d}{2} r^{d-1} - \int_{|T|=r} kr \frac{h(T)}{r} d\sigma \\ &= \frac{c_d}{2} r^{d-1} \left(1 - \frac{2k}{c_d} \int_{|T|=r} \frac{h(T)}{|T|^{d-1}} d\sigma \right). \end{aligned}$$

This proves Lemma 5.5.

Proof of Lemma 5.4 We now know that

$$S(E_r) \leq \frac{c_d}{2} r^{d-1} (1 - k(r)) \quad (5.3.4)$$

where, by (5.3.2),

$$\int_0^1 \frac{k(r)}{r} dr = \infty. \quad (5.3.5)$$

We follow an argument used by Rippon [29] in which he uses the Carleman method to obtain upper bounds for harmonic measure.

Let D_1 be the domain $D \cap B(0, 1)$ and let $\omega(X, y)$ denote the harmonic measure with respect to D_1 of $D \cap \{|X|^2 + y^2 = 1\}$. It is enough to prove that

$$\frac{\omega(0, y)}{y} \rightarrow 0 \text{ as } y \rightarrow 0^+. \quad (5.3.6)$$

Following (1.2.12), the Carleman mean of $\omega(X, y)$ is given by

$$m(r) = \left\{ \frac{1}{c_d r^{d-1}} \int_{E_r} \omega(X, y)^2 d\sigma \right\}^{\frac{1}{2}}, \quad 0 < r < 1.$$

Then, by Theorem 1.F,

$$r \frac{d}{dr} \{\log A(r)\} \geq 2\alpha(r) + d - 2, \quad 0 < r < 1 \quad (5.3.7)$$

where

$$A(r) = r \frac{d}{dr} (m(r)^2 r^{d-2}).$$

Theorem 1.D states that if E is a set on $\{|X|^2 + y^2 = 1\}$ and E^* is a spherical cap with the same surface area as E , then

$$\alpha_E \geq \alpha_{E^*}$$

and Theorem 1.E then gives that

$$\alpha_{E^*} \geq 2 \left(1 - \frac{S(E^*)}{c_d} \right)$$

where $S(E^*)$ is the surface area of E^* and hence of E . So for any open set E on $\{|X|^2 + y^2 = r^2\}$,

$$\alpha_E \geq 2 \left(1 - \frac{S(E)}{r^{d-1} c_d} \right).$$

According to (5.3.4)

$$\begin{aligned}\alpha(r) &\geq 2 - \frac{2}{r^{d-1}c_d} \left\{ \frac{r^{d-1}c_d}{2} (1 - k(r)) \right\} \\ &= 1 + k(r).\end{aligned}$$

This, with (5.3.7), implies that

$$r \frac{d}{dr} \{\log A(r)\} \geq 2 + 2k(r) + d - 2$$

i.e.

$$\frac{d}{dr} \{\log A(r)\} \geq \frac{d}{r} + 2 \frac{k(r)}{r}. \quad (5.3.8)$$

Integrating (5.3.8) yields

$$\frac{A(r_1)}{r_1^d} \leq \frac{A(r_2)}{r_2^d} \exp \left(-2 \int_{r_1}^{r_2} \frac{k(r)}{r} dr \right). \quad (5.3.9)$$

Now,

$$\frac{A(r)}{r^d} = (d-2) \left\{ \frac{m(r)}{r} \right\}^2 + \frac{2m(r)m'(r)}{r} \quad (5.3.10)$$

and

$$m(r)m'(r) = \frac{1}{c_d r^{d-1}} \int_{E_r} \omega \frac{\partial \omega}{\partial r} d\sigma = \frac{1}{c_d r^{d-1}} \int_{D_1} |\nabla \omega_1|^2 d(X, y). \quad (5.3.11)$$

Hence $m'(r) \geq 0$ while $0 \leq m(r) \leq 1$, and so the set of r in $(0, 1)$ where $m'(r) > 3$ can have length at most $\frac{1}{3}$. Thus there exists $r_0 > \frac{1}{2}$ such that $m'(r_0) \leq 3$ and we deduce from (5.3.10) that

$$\frac{A(r_0)}{r_0^d} \leq 4(d+1).$$

It follows, by (5.3.9), that

$$\frac{A(r)}{r^d} \leq 4(d+1) \exp \left(-2 \int_r^{r_0} \frac{k(r)}{r} dr \right), 0 < r \leq \frac{1}{2}$$

and hence, by (5.3.10), that

$$\frac{m(r)}{r} \leq \text{const.} \exp \left(- \int_r^{r_0} \frac{k(r)}{r} dr \right) \quad (5.3.12)$$

for a constant depending only on d .

Finally, by (1.2.15),

$$\omega(0, \frac{r}{2}) \leq 3 \cdot 2^{d-2} m(r),$$

so that, by (5.3.12),

$$\frac{\omega(0, y)}{y} \leq \text{const.} \exp \left(- \int_{2y}^{r_0} \frac{k(r)}{r} dr \right), 0 < y \leq \frac{1}{4}.$$

It then follows from (5.3.5) that

$$\frac{\omega(0, y)}{y} \rightarrow 0.$$

This completes the proof of Lemma 5.4.

Suppose now that D is given by (5.1.1) where $h(T)$ satisfies (5.1.2) and (5.1.3). The following geometric lemma is an essential step in the proof of Theorem 4.1.

Lemma 5.6 *There is a sequence of balls $\Delta_n = B(X_n, r_n)$ contained in D for which*

(i) $|X_n|$ is decreasing and the balls $B(X_n, 2r_n)$ are mutually disjoint,

(ii)

$$\sum_{n=1}^{\infty} \left(\frac{r_n}{|X_n|} \right)^d = \infty, \quad (5.3.13)$$

but, for each n ,

$$\sum_{k=1}^{n-1} \left(\frac{r_k}{|X_k|} \right)^d < \frac{1}{2K_1} \log \frac{1}{|X_n|} \quad (5.3.14)$$

where K_1 is a constant given by (5.4.5)

(iii) and, for each n ,

$$\int_{|T|<1} \frac{h^+(T)}{|T - X_n|^d} dT \leq 2^{d+1} \int_{|T|<1} \frac{h^+(T)}{|T|^d} dT. \quad (5.3.15)$$

Proof of Lemma 5.6 First of all we choose the balls Δ_n so that part (i) and (5.3.13) hold.

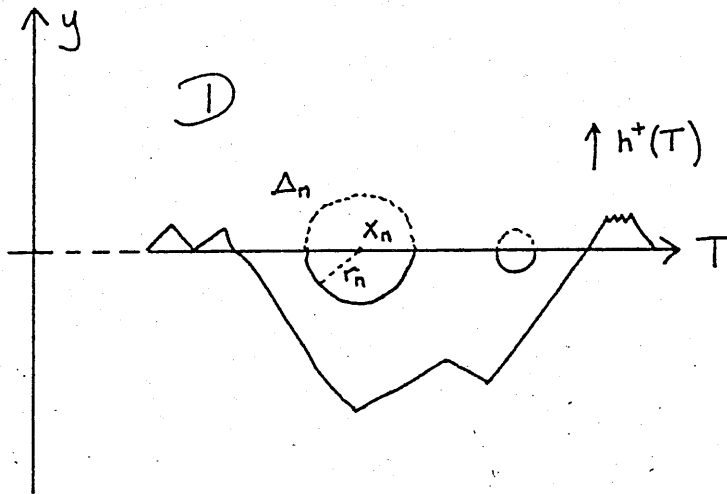


Figure 5.2 A typical ball Δ_n

Since $h(X)$ is Lipschitz we have, for $|X_1|, |X_2| < 1$,

$$|h(X_1) - h(X_2)| \leq |X_1 - X_2|.$$

If $X_0 \in F$ and a is positive, let $C(X_0, a)$ be the cone

$$C(X_0, a) = \{(X, y) : y > -(a - |X - X_0|)\}.$$

To begin with, we choose a sequence of cones $C(X_n, a_n)$ each of which is contained in D and later the balls in the statement of the lemma are fitted inside these cones. In choosing the cones we also define a sequence of $(d-1)$ -dimensional balls B_n contained in F . Supposing that the cones $C(X_n, a_n)$ and the balls $B_n, n = 1, 2, \dots, N-1$ have already been chosen, we choose $C(X_N, a_N)$ and B_N in the following way. First put

$$a_N = \max\{h^-(X) : X \notin \bigcup_{i=1}^{N-1} B_i\}$$

and suppose that

$$h^-(X_N) = a_N.$$

Since $h(X)$ has Lipschitz constant 1, it follows that

$$C(X_N, a_N) \subseteq D.$$

Now let B_N be the $(d-1)$ -ball $|X - X_N| < a_N$. If $X \in B_N$, then

$$h^-(X) - h^-(X_N) \leq |h^-(X) - h^-(X_N)| \leq |X - X_N| \leq a_N$$

and so

$$h^-(X) \leq 2a_N.$$

Thus,

$$\begin{aligned} \int_{B_N} \frac{h^-(T)}{|T|^d} dT &\leq 2a_N \int_{B_N} \frac{dT}{|T|^d} \\ &\leq c \left(\frac{a_N}{|X_N|} \right)^d, \end{aligned} \quad (5.3.16)$$

where the constant c depends only on d .

Now consider the $(d-1)$ -dimensional balls B'_n where

$$B'_n = \{X \in F : |X - X_n| < \frac{1}{2}a_n\}.$$

Note that these balls are mutually disjoint. For if $n_2 > n_1$, then $X_{n_2} \notin B_{n_1}$ by construction and moreover $a_{n_2} \leq a_{n_1}$. Thus,

$$|X_{n_1} - X_{n_2}| \geq a_{n_1} \geq \frac{1}{2}a_{n_1} + \frac{1}{2}a_{n_2}$$

and so the balls B'_{n_1} and B'_{n_2} do not intersect.

It follows that a_n tends to zero, because the disjoint balls $|X - X_n| < \frac{1}{2}a_n$ all lie in the ball $|X| < 2$. So if X_0 is a point of F for which $h^-(X_0) > 0$ then, for some n_0 , $a_{n_0} < h(X_0)$ and it follows from the definition of a_{n_0} that $X_0 \in B_i$ for some i , $1 \leq i \leq n_0 - 1$. Hence, by (5.1.3),

$$\int_{\bigcup_1^\infty B_n} \frac{h^-(T)}{|T|^d} dT = \infty. \quad (5.3.17)$$

From (5.3.16) and (5.3.17) it follows that

$$\sum_{n=1}^\infty \left(\frac{a_n}{|X_n|} \right)^d = \infty. \quad (5.3.18)$$

It may happen that the centres X_n of the balls B_n accumulate at points other than the origin. This is a problem if we now try to reorder the

X_n in terms of decreasing modulus. To overcome this, we consider the contribution to the sum in (5.3.18) from each of the annuli

$$A_n = \{X : 2^{-n-1} < |X| \leq 2^{-n}\}, n = 0, 1, 2, \dots$$

in turn. If there are infinitely many X_n in a particular annulus A_{n_0} , we can choose finitely many of these $X_n \in A_{n_0}$ so that they contribute at least half as much as did the original X_n 's to the sum in (5.3.18). Thus it may be assumed that there are only finitely many X_n in each annulus A_n and that (5.3.18) holds. It may be assumed, therefore, that $|X_n|$ is decreasing.

Noting that

$$B(X_n, 2r_n) \subseteq C(X_n, \frac{1}{2}a_n)$$

where $2r_n = a_n/2\sqrt{2}$ and using the disjointness of the balls B'_n , we obtain a sequence of balls which satisfies (i) and (5.3.13).

We now turn to part (iii). It suffices to show that (5.3.15) holds apart from a subsequence of X_n for which the sum in (5.3.13) is finite. Put

$$I = \int_{|T|<1} \frac{h^+(T)}{|T|^d} dT,$$

and write $V(X, y)$ for the harmonic function in the half-space H with boundary values on F given by $V(X) = h^+(X)$ when $|X| < 1$ and zero otherwise, i.e. (see (1.2.9))

$$V(X, y) = \frac{2}{c_d} \int_{|T|<1} \frac{y h^+(T)}{(|X - T|^2 + y^2)^{\frac{d}{2}}} dT. \quad (5.3.19)$$

Then

$$\frac{2}{c_d} \int_{|T|<1} \frac{h^+(T)}{|T - X_n|^d} dT = \frac{\partial V}{\partial y} \Big|_{X_n} = \lim_{y \rightarrow 0^+} \frac{V(X_n, y)}{y}, \quad (5.3.20)$$

since $V(X_n, 0) = 0$.

First of all we show that if, for a certain n ,

$$\frac{\partial V}{\partial y} \Big|_{X_n} = \frac{2}{c_d} \int_{|T|<1} \frac{h^+(T)}{|T - X_n|^d} dT \geq \frac{2^{d+2}}{c_d} I, \quad (5.3.21)$$

then

$$\frac{V(X, y)}{y} \geq \frac{4}{c_d} I, \quad (5.3.22)$$

for $|X - X_n| < r_n, 0 < y < r_n$.

Suppose then that $|X - X_n| < r_n, 0 < y < r_n$ and that $|T - X_n| > 2r_n$.

Then

$$\begin{aligned}
 (|X - T|^2 + y^2)^{\frac{1}{2}} &\leq |X - T| + y \\
 &\leq |T - X_n| + |X_n - X| + y \\
 &\leq 2r_n + |T - X_n| \\
 &\leq 2|T - X_n|.
 \end{aligned}$$

Since the ball $B(X_n, 2r_n)$ is contained in D , $h^+(T) = 0$ for $|T - X_n| < 2r_n$. Thus,

$$\begin{aligned}
 \frac{V(X, y)}{y} &= \frac{2}{c_d} \int_{|T| < 1} \frac{h^+(T)}{(|X - T|^2 + y^2)^{\frac{d}{2}}} dT \\
 &\geq \frac{2}{2^d c_d} \int_{|T| < 1} \frac{h^+(T)}{|T - X_n|^d} dT \\
 &\geq \frac{4}{c_d} I
 \end{aligned}$$

by (5.3.21). This establishes (5.3.22).

We see from the representation (5.3.19) for $V(X, y)$ that for a fixed X , $V(X, y)/y$ decreases strictly with increasing y . This allows us to define a nonnegative function $f_1(X)$ on F by the equation

$$V(X, f_1(X)) - \frac{4}{c_d} I f_1(X) = 0. \quad (5.3.23)$$

Put

$$H(X, y) = \frac{4}{c_d} I y - V(X, y)$$

so that $H(X, y)$ is positive and harmonic in the domain

$$D_1 = \{(X, y) : y > f_1(X)\}$$

and vanishes at all finite boundary points of D_1 . Moreover,

$$\begin{aligned}
 \lim_{y \rightarrow 0^+} \frac{H(0, y)}{y} &= \frac{4}{c_d} I - \frac{2}{c_d} \int_{|T| < 1} \frac{h^+(T)}{|T|^d} dT \\
 &= \frac{2}{c_d} I \\
 &> 0.
 \end{aligned} \quad (5.3.24)$$

We deduce from (5.3.21) and (5.3.22) that the surface $(X, f_1(X))$ lies *above* those cylinders

$$\{(X, y) : |X - X_n| < r_n, 0 < y < r_n\}$$

which correspond to balls Δ_n for which

$$\int_{|T|<1} \frac{h^+(T)}{|T - X_n|^d} dT > 2^{d+1} I. \quad (5.3.25)$$

Let Δ_{n_k} be the subsequence of balls Δ_n whose centres $(X_n, 0)$ satisfy the inequality (5.3.25). We wish to show that

$$\sum_k \left(\frac{r_{n_k}}{|X_{n_k}|} \right)^d < \infty. \quad (5.3.26)$$

Put

$$f_2(X) = \begin{cases} r_{n_k} - |X_{n_k} - X|, & |X_{n_k} - X| < r_{n_k} \text{ some } k, \\ 0, & \text{otherwise,} \end{cases}$$

and put

$$D_2 = \{(X, y) : y > f_2(X)\}.$$

Thus $D_1 \subseteq D_2$ since $f_2(X) \leq f_1(X)$ and so we may extend $H(X, y)$ to be subharmonic in D_2 by putting $H(X, y) = 0$ for $(X, y) \in D_2 \setminus D_1$. Noting that $4Iy/c_d$ is a harmonic majorant of H in D_2 , let $H^*(X, y)$ be the least harmonic majorant of $H(X, y)$ in D_2 , so that $H^*(X, y)$ is a positive harmonic function in D_2 which vanishes at finite boundary points of D_2 . If the sum in (5.3.26) is not finite we conclude that

$$\int_{|T|<1} \frac{f_2(T)}{|T|^d} dT = \infty.$$

and hence by Lemma 5.4 that

$$\frac{H^*(0, y)}{y} \rightarrow 0 \text{ as } y \rightarrow 0^+,$$

which contradicts (5.3.24).

It remains to establish (5.3.14). Write

$$a_n = K_1 \left(\frac{r_n}{|X_n|} \right)^d.$$

We select a subsequence n_k such that

$$\sum_{k=1}^{\infty} a_{n_k} = \infty, \quad (5.3.27)$$

but

$$\sum_{k=1}^{p-1} a_{n_k} < \frac{1}{2} \log \frac{1}{|X_{n_p}|}, \quad p > 2. \quad (5.3.28)$$

To do this we define n_p inductively to be the first integer such that (5.3.28) holds and $n_p > n_{p-1}$. Suppose that (5.3.27) is false. Then (5.3.28) holds for $n_p = n_{p-1} + 1$ if p is large, since $X_n \rightarrow 0$. Thus $n_p = n_{p-1} + 1$ for large p and (5.3.27) holds after all. This completes the proof of Lemma 5.6.

5.4 Constructing the function of Lemma 5.3

The following constants will be needed in due course. Write

$$I_1 = \int_{|T|<1} \frac{1 - |T|}{(1 + |T|^2)^{\frac{d}{2}}} dT \quad (5.4.1)$$

and

$$I_2 = c_{d-1} \int_0^1 \frac{1 - t^2}{(1 + t^2)^{\frac{d}{2}}} t^{d-2} dt. \quad (5.4.2)$$

We need to construct a function $U(X, y)$ in D which has the properties described in Lemma 5.3, namely $U(X, y)$ is nonpositive on the boundary of D , is subharmonic near zero and (5.2.6) holds. In view of the local nature of Lemma 5.3, we may assume that the finite integral in (5.1.2) is small. In fact, we assume that

$$\int_{|T|<1} \frac{h^+(T)}{|T|^d} dT < \frac{I_1}{8.48^d}. \quad (5.4.3)$$

First we obtain from Lemma 5.6 a sequence of balls $\{\Delta_n\} = B(X_n, r_n)$ which satisfies the conclusions of that lemma. We then put

$$R = (D \cap H) \cup (\cup_n \Delta_n),$$

so that R is a subdomain of D . The function $U(X, y)$ which we construct is defined in R and is nonpositive on the boundary of R . Thus by taking

the maximum of $U(X, y)$ and 0 in R and 0 in $D \setminus R$ we obtain the required subharmonic function in D .

Next we define a sequence of numbers $\{\mu_n\}_1^\infty$ by setting $\mu_1 = 1$ and, for $N = 2, 3, \dots$, by the recurrence relation

$$\frac{\mu_N}{r_N} = K_1 \sum_{n=1}^{N-1} \frac{\mu_n r_n^{d-1}}{|X_n|^d}, \quad (5.4.4)$$

where

$$K_1 = \min \left(1, \frac{2^{d+1} I_2}{I_2 + 4d6^d c_d} \right). \quad (5.4.5)$$

Lemma 5.7 *If $N \geq 3$, we have*

$$\frac{\mu_N}{r_N} \geq \frac{\mu_2}{r_2} \exp \left\{ \frac{K_1}{2} \sum_{n=2}^{N-1} \left(\frac{r_n}{|X_n|} \right)^d \right\}. \quad (5.4.6)$$

Also,

$$\mu_n \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (5.4.7)$$

Proof On solving the recurrence relation (5.4.4) we obtain, for $N \geq 3$,

$$\frac{\mu_N}{r_N} = \frac{\mu_2}{r_2} \prod_{n=2}^{N-1} \left[1 + K_1 \left(\frac{r_n}{|X_n|} \right)^d \right]. \quad (5.4.8)$$

Since $x/2 \leq \log(1+x) < x$ for $0 \leq x \leq 1$ and $K_1 \leq 1$, we have

$$\begin{aligned} \frac{\mu_N}{r_N} &= \frac{\mu_2}{r_2} \exp \left\{ \sum_{n=2}^{N-1} \log \left[1 + K_1 \left(\frac{r_n}{|X_n|} \right)^d \right] \right\} \\ &\geq \frac{\mu_2}{r_2} \exp \left\{ \frac{K_1}{2} \sum_{n=2}^{N-1} \left(\frac{r_n}{|X_n|} \right)^d \right\}, \end{aligned}$$

which is (5.4.6), and we have

$$\frac{\mu_N}{r_N} \leq \frac{\mu_2}{r_2} \exp \left\{ K_1 \sum_{n=2}^{N-1} \left(\frac{r_n}{|X_n|} \right)^d \right\}. \quad (5.4.9)$$

We now deduce from (5.3.14) and (5.4.9) that

$$\frac{\mu_n}{r_n} \leq \frac{\mu_2}{r_2} |X_n|^{-\frac{1}{2}} < \frac{\mu_2}{r_2} r_n^{-\frac{1}{2}},$$

i.e.

$$\mu_n < \frac{\mu_2}{r_2} r_n^{\frac{1}{2}},$$

so that $\mu_n \rightarrow 0$. This completes the proof of Lemma 5.7.

Write $\Gamma_n = \Delta_n \cap F$ and let P_n denote the Poisson kernel of Δ_n with singularity at (X_n, r_n) , normalised so that $P_n(X_n) = 1$. Note that, with I_2 given by (5.4.2),

$$\int_{\Gamma_n} P_n(T) dT = I_2 r_n^{d-1}. \quad (5.4.10)$$

We begin the construction of $U(X, y)$ by defining

$$U(X, y) = \mu_n P_n(X, y), \quad (5.4.11)$$

for $(X, y) \in \Delta_n \cap \{y \leq 0\}$. Thus on F we have a function $u_1(X)$ where

$$u_1(X) = \begin{cases} \mu_n P_n(X), & X \in \Gamma_n, \\ 0, & \text{otherwise,} \end{cases} \quad (5.4.12)$$

and then, for $(X, y) \in H$, we define

$$u_1(X, y) = \frac{2}{c_d} \int_F \frac{y u_1(T)}{(|X - T|^2 + y^2)^{\frac{d}{2}}} dT. \quad (5.4.13)$$

Since by (5.4.7) the numbers μ_n tend to zero, $u_1(X)$ is continuous on F and $u_1(X, y)$ is positive and harmonic in H . Now, $u_1(X, y)$ is positive at points $(X, h^+(X))$, where $h^+(X) > 0$, on the boundary of R . So to compensate for this we subtract from $u_1(X, y)$ a positive harmonic function $u_2(X, y)$ which is the extension into H of the boundary function

$$u_2(X) = \alpha(X) h^+(X) \quad (5.4.14)$$

where

$$\alpha(X) = K_2 \sum_{n=1}^{\infty} \frac{\mu_n r_n^{d-1}}{|X - X_n|^d} \quad (5.4.15)$$

and

$$K_2 = \frac{2^{2d} I_2}{I_1}. \quad (5.4.16)$$

Note that $u_2(X)$ is bounded on F . In fact, consider the range $2^p h^+(X) \leq |X - X_n| < 2^{p+1} h^+(X)$, $p \geq 1$, and denote the corresponding sum by \sum_p . Then, since the balls Δ_n are disjoint,

$$\begin{aligned} \sum_p &\leq K_2 M h^+(X) \frac{\sum r_n^{d-1}}{(2^p h^+(X))^d} \\ &\leq c h^+(X) \frac{(2^{p+2} h^+(X))^{d-1}}{(2^p h^+(X))^d} \\ &\leq c 2^{-p}, \end{aligned}$$

where $\mu_n \leq M$ by (5.4.7). Hence,

$$u_2(X) \leq \sum_{p=1}^{\infty} \sum_p \leq 2c,$$

and $u_2(X)$ is uniformly bounded on F .

Thus we complete the construction of $U(X, y)$ in R by putting

$$U(X, y) = u_1(X, y) - u_2(X, y), \quad (5.4.17)$$

for $(X, y) \in H$.

5.5 Proof of Theorem 4.1

As noted earlier, we need to show that the function $U(X, y)$ constructed in the previous section is nonpositive on the boundary of R , is subharmonic near 0 and that (5.2.6) holds. We make a start on this task in the next lemma.

Lemma 5.8 *The function $U(X, y)$ defined by (5.4.17) is nonpositive on the boundary of R .*

Proof Other than at points $(X, h^+(X))$ where $h^+(X) > 0$, we have $U = 0$ on ∂R by construction. Suppose then that $h^+(X) > 0$. We need an estimate for $u_1(X, h^+(X))$. From (5.4.12) and (5.4.13),

$$\begin{aligned} u_1(X, h^+(X)) &= \frac{2}{c_d} \int_F \frac{h^+(X) u_1(T)}{(|X - T|^2 + h^+(X)^2)^{\frac{d}{2}}} dT \\ &= \frac{2}{c_d} h^+(X) \sum_{n=1}^{\infty} \int_{\Gamma_n} \frac{\mu_n P_n(T)}{(|X - T|^2 + h^+(X)^2)^{\frac{d}{2}}} dT. \end{aligned}$$

For $T \in \Gamma_n$, we have

$$(|X - T|^2 + h^+(X)^2)^{\frac{1}{2}} \geq |X - T| \geq |X - X_n| - r_n.$$

By Lemma 5.6 (i), $2r_n < |X - X_n|$ and so

$$(|X - T|^2 + h^+(X)^2)^{\frac{1}{2}} \geq \frac{1}{2}|X - X_n|.$$

Therefore,

$$\begin{aligned} u_1(X, h^+(X)) &\leq \frac{2}{c_d} h^+(X) \sum_{n=1}^{\infty} \frac{2^d}{|X - X_n|^d} \int_{\Gamma_n} \mu_n P_n(T) dT \\ &= K_3 h^+(X) \sum_{n=1}^{\infty} \frac{\mu_n r_n^{d-1}}{|X - X_n|^d} \\ &= \frac{K_3}{K_2} u_2(X), \end{aligned} \tag{5.5.1}$$

where, by (5.4.10),

$$K_3 = \frac{2^{d+1}}{c_d} I_2.$$

We next estimate $u_2(X, h^+(X))$, where $h^+(X) > 0$ and $u_2(X, y)$ is the extension into H of the boundary values given by (5.4.14) and (5.4.15). We have,

$$\begin{aligned} u_2(X, h^+(X)) &= \frac{2}{c_d} \int_F \frac{h^+(X) u_2(T)}{(|X - T|^2 + h^+(X)^2)^{\frac{d}{2}}} dT \\ &\geq \frac{2}{c_d} \int_{|X-T| < h^+(X)} \frac{h^+(X) u_2(T)}{(|X - T|^2 + h^+(X)^2)^{\frac{d}{2}}} dT. \end{aligned}$$

Note that because of the Lipschitz condition on h^+ , we have $h^+(T) \geq h^+(X) - |T - X|$ for $|T - X| < h^+(X)$. Thus, for T in this range,

$$u_2(T) \geq \alpha^*(X)(h^+(X) - |T - X|),$$

where

$$\alpha^*(X) = \min\{\alpha(T) : |T - X| < h^+(X)\}.$$

So,

$$\begin{aligned} u_2(X, h^+(X)) &\geq \frac{2h^+(X)}{c_d} \alpha^*(X) \int_{|T-X| < h^+(X)} \frac{h^+(X) - |T - X|}{(|X - T|^2 + h^+(X)^2)^{\frac{d}{2}}} dT \\ &= K_4 h^+(X) \alpha^*(X), \end{aligned} \tag{5.5.2}$$

where

$$K_4 = \frac{2}{c_d} I_1$$

and I_1 is given by (5.4.1). It remains to estimate $\alpha^*(X)$.

If $|X - T| < h^+(X)$, then for $n = 1, 2, \dots$,

$$|T - X_n| \leq |T - X| + |X - X_n| \leq h^+(X) + |X - X_n| \leq 2|X - X_n|$$

so that by (5.4.15)

$$\alpha^*(X) \geq 2^{-d} \alpha(X). \quad (5.5.3)$$

Now it follows from (5.5.2) and (5.5.3) that

$$u_2(X, h^+(X)) \geq K_4 2^{-d} \alpha(X) h^+(X) = K_4 2^{-d} u_2(X). \quad (5.5.4)$$

Comparing the estimates (5.5.1) and (5.5.4) we see that

$$u_2(X, h^+(X)) \geq u_1(X, h^+(X))$$

and so $U(X, y)$ is nonpositive at $(X, h^+(X))$. The only remaining point to check is the origin. To dispose of this case note that $U(X, y) \leq u_1(X, y)$ and that $u_1(X, y)$ is continuous at the origin with $u_1(0) = 0$ since $\mu_n \rightarrow 0$ as $n \rightarrow \infty$. This completes the proof of Lemma 5.8.

For each X in R we use (5.2.3) and (5.2.4) with $u = U$ as definitions of $D_+U(X)$ and $D_-U(X)$ respectively. Estimates for $D_+U(X)$ and $D_-U(X)$ are obtained in the next lemma and we show that (5.2.5) holds for all X in R close to 0. Then Lemma 5.2 tells us that $U(X, y)$ is subharmonic near 0.

Lemma 5.9 *If $X \in \Gamma_N$ then*

(i)

$$D_-U(X) \leq \frac{d\mu_N}{r_N}, \quad (5.5.5)$$

(ii)

$$D_+U(X) = \frac{\partial u_1}{\partial y} \Big|_{(X,0)} - \frac{\partial u_2}{\partial y} \Big|_{(X,0)} \quad (5.5.6)$$

where

$$\frac{\partial u_1}{\partial y} \Big|_{(X,0)} \geq K_5 \sum_{n=1, n \neq N}^{\infty} \frac{\mu_n r_n^{d-1}}{|X_n - X_N|^d} - \frac{d\mu_N}{r_N} \quad (5.5.7)$$

and

$$\frac{\partial u_2}{\partial y} \Big|_{(X,0)} \leq \frac{K_5}{2} \sum_{n=1, n \neq N}^{\infty} \frac{\mu_n r_n^{d-1}}{|X_n - X_N|^d} + \frac{K_7 \mu_N}{r_N}. \quad (5.5.8)$$

Here

$$K_5 = \frac{2}{c_d} \left(\frac{2}{3}\right)^d I_2 \text{ and } K_7 = \frac{I_2}{2 \cdot 6^d c_d}.$$

We deduce that $U(X, y)$ is subharmonic in a neighbourhood of 0.

Proof The inequality (5.5.5) is a consequence of (5.4.11), (5.2.4) and Lemma 5.1.

The equation (5.5.6) is obtained on differentiating (5.4.17) according to (5.2.3), with u replaced by U .

First of all, we estimate from below the contribution to $D_+ U(X)$ from u_1 . The contribution from $P_N(T)$ on Γ_N itself, works against us. However, by the maximum principle, the derivative does not exceed what it would be if D were replaced by the ball Δ_N , i.e. $d\mu_N/r_N$ as above. Thus, if $X \in \Gamma_N$, then

$$\begin{aligned} \frac{\partial u_1}{\partial y} \Big|_{(X,0)} &\geq \frac{2}{c_d} \int_{F \setminus \Gamma_N} \frac{u_1(T)}{|X - T|^d} dT - \frac{d\mu_N}{r_N} \\ &= \frac{2}{c_d} \sum_{n=1, n \neq N}^{\infty} \int_{\Gamma_n} \frac{\mu_n P_n(T)}{|X - T|^d} dT - \frac{d\mu_N}{r_N}. \end{aligned}$$

If $T \in \Gamma_n, n \neq N$, then

$$|X - T| \leq |X_n - X_N| + r_n + r_N \leq \frac{3}{2} |X_n - X_N|,$$

by Lemma 5.6 (i). Therefore, by (5.4.10),

$$\frac{\partial u_1}{\partial y} \Big|_{(X,0)} \geq K_5 \sum_{n=1, n \neq N}^{\infty} \frac{\mu_n r_n^{d-1}}{|X_n - X_N|^d} - \frac{d\mu_N}{r_N}$$

which is (5.5.7).

Next, if $X \in \Gamma_N$, then the contribution to $D_+ U(X)$ from u_2 is

$$\begin{aligned} \frac{\partial u_2}{\partial y} \Big|_{(X,0)} &= \frac{2}{c_d} \int_F \frac{u_2(T)}{|X - T|^d} dT \\ &= \frac{2}{c_d} \int_F \frac{\alpha(T) h^+(T)}{|X - T|^d} dT. \end{aligned}$$

Since $|X - X_N| < r_N$, we have $|X - T| > |X_N - T| - r_N$. Also, if $h^+(T) > 0$ then $2r_N < |T - X_N|$ and so $|X - T| > \frac{1}{2}|X_N - T|$. Therefore, by (5.4.15),

$$\begin{aligned}
\frac{\partial u_2}{\partial y} \Big|_{(X,0)} &\leq \frac{2^{d+1}}{c_d} \int_F \frac{\alpha(T)h^+(T)}{|X_N - T|^d} dT \\
&= \frac{2^{d+1}}{c_d} \int_F \left[K_2 \sum_{n=1}^{\infty} \frac{\mu_n r_n^{d-1}}{|T - X_n|^d} \right] \frac{h^+(T)}{|X_N - T|^d} dT \\
&= \frac{2^{d+1} K_2}{c_d} \sum_{n=1}^{\infty} \int_F \frac{h^+(T)}{|X_N - T|^d} \cdot \frac{\mu_n r_n^{d-1}}{|T - X_n|^d} dT \\
&= K_6 \sum_{n=1}^{\infty} \mu_n r_n^{d-1} \left(\int_F \frac{h^+(T) dT}{|X_N - T|^d |T - X_n|^d} \right), \quad (5.5.9)
\end{aligned}$$

where, by (5.4.16),

$$K_6 = \frac{2^{3d+1} I_2}{c_d I_1}.$$

It remains to estimate

$$\int_F \frac{h^+(T) dT}{|X_N - T|^d |T - X_n|^d}. \quad (5.5.10)$$

Suppose first that $n \neq N$. Consider

$$J = \int_{|T|<1} \frac{|X_N - X_n|^d}{|X_N - T|^d |X_n - T|^d} h^+(T) dT.$$

Since,

$$\frac{|X_N - X_n|}{|X_N - T| |X_n - T|} \leq 2 \max \left\{ \frac{1}{|X_N - T|}, \frac{1}{|X_n - T|} \right\},$$

it follows that

$$\begin{aligned}
J &\leq 2^d \int_{|T|<1} \left\{ \frac{1}{|X_N - T|^d} + \frac{1}{|X_n - T|^d} \right\} h^+(T) dT \\
&\leq 2^{d+1} \max_n \left\{ \int_{|T|<1} \frac{h^+(T)}{|T - X_n|^d} dT \right\} \\
&\leq 2^{d+1} \cdot 2^{d+1} \int_{|T|<1} \frac{h^+(T)}{|T|^d} dT
\end{aligned}$$

by Lemma 5.6 (iii). It then follows from (5.4.3) that

$$\begin{aligned}
J &\leq 2^{d+1} \cdot 2^{d+1} \cdot \frac{I_1}{8.48^d} \\
&= \frac{I_1}{2.12^d} \\
&= \frac{K_5}{2K_6}.
\end{aligned}$$

The case $n = N$ remains. By Lemma 5.6 (iii) and since $h^+(T) > 0$ implies that $|X_N - T| > 2r_N$, we have

$$\begin{aligned} \int_{|T|<1} \frac{h^+(T)dT}{|X_N - T|^{2d}} &\leq \left(\frac{1}{2r_N}\right)^d \int_{|T|<1} \frac{h^+(T)dT}{|X_N - T|^d} \\ &\leq \left(\frac{1}{2r_N}\right)^d \left\{ 2^{d+1} \int_{|T|<1} \frac{h^+(T)}{|T|^d} dT \right\} \\ &\leq \left(\frac{1}{2r_N}\right)^d \left\{ 2^{d+1} \frac{I_1}{8.48^d} \right\} \\ &= \left(\frac{I_1}{4.48^d}\right) \frac{1}{r_N^d}. \end{aligned}$$

Thus,

$$K_6 \mu_N r_N^{d-1} \int_F \frac{h^+(T)dT}{|X_N - T|^{2d}} \leq K_7 \frac{\mu_N}{r_N},$$

which completes the proof of (5.5.8).

It follows from Lemma 5.2 that $U(X, y)$ will be subharmonic close to 0 if

$$D_- U(X) \leq D_+ U(X),$$

when $X \in R$ and X is close to 0. Thus the estimates (5.5.5), (5.5.6), (5.5.7) and (5.5.8) give a criterion for $U(X, y)$ to be subharmonic near 0, namely that for all large N ,

$$(2d + K_7) \frac{\mu_N}{r_N} \leq \frac{K_5}{2} \sum_{n=1, n \neq N}^{\infty} \frac{\mu_n r_n^{d-1}}{|X_n - X_N|^d}. \quad (5.5.11)$$

Since $|X_n - X_N| \leq 2|X_n|$ for $n < N$, (5.5.11) holds if

$$\frac{\mu_N}{r_N} \leq \frac{K_5}{(2d + K_7)2^{d+1}} \sum_{n=1}^{N-1} \frac{\mu_n r_n^{d-1}}{|X_n|^d}, \quad N = 2, 3, \dots,$$

which is the case by the definition (5.4.4). This completes the proof of Lemma 5.9.

We have shown that the function $U(X, y)$ is subharmonic near zero and is nonpositive at finite boundary points of the domain R . To complete the proof of Lemma 5.3 we prove (5.2.6).

Lemma 5.10 *The function $U(X, y)$ satisfies (5.2.6) in that*

$$\frac{U(0, y)}{y} \rightarrow \infty \text{ as } y \rightarrow 0^+. \quad (5.5.12)$$

Proof. To begin with, (5.4.12) and (5.4.13) give

$$\begin{aligned} \frac{u_1(0, y)}{y} &= \frac{2}{c_d} \int_F \frac{u_1(T)}{(|T|^2 + y^2)^{\frac{d}{2}}} dT \\ &= \frac{2}{c_d} \sum_{n=1}^{\infty} \int_{\Gamma_n} \frac{\mu_n P_n(T)}{(|T|^2 + y^2)^{\frac{d}{2}}} dT. \end{aligned}$$

If $|X_n| \geq y$ and $|T - X_n| < r_n$, then

$$(|T|^2 + y^2)^{\frac{1}{2}} \leq |T| + y \leq 3|X_n|,$$

and if $|X_n| < y$, then

$$(|T|^2 + y^2)^{\frac{1}{2}} \leq 3y.$$

Thus, if we put

$$m(n, y) = \max\{|X_n|, y\},$$

then

$$\frac{u_1(0, y)}{y} \geq K_8 \sum_{n=1}^{\infty} \frac{\mu_n r_n^{d-1}}{m(n, y)^d}, \quad (5.5.13)$$

where, by (5.4.10),

$$K_8 = \frac{2I_2}{3^d c_d}.$$

Next, by (5.4.14) and (5.4.15),

$$\begin{aligned} \frac{u_2(0, y)}{y} &= \frac{2}{c_d} \int_F \frac{u_2(T)}{(|T|^2 + y^2)^{\frac{d}{2}}} dT \\ &= \frac{2}{c_d} \int_F \frac{\alpha(T) h^+(T)}{(|T|^2 + y^2)^{\frac{d}{2}}} dT \\ &= \frac{2}{c_d} \int_F \left[K_2 \sum_{n=1}^{\infty} \frac{\mu_n r_n^{d-1}}{|T - X_n|^d} \right] \frac{h^+(T)}{(|T|^2 + y^2)^{\frac{d}{2}}} dT \\ &= K_9 \sum_{n=1}^{\infty} \mu_n r_n^{d-1} \left(\int_F \frac{h^+(T)}{|T - X_n|^d (|T|^2 + y^2)^{\frac{d}{2}}} dT \right), \quad (5.5.14) \end{aligned}$$

where, by (5.4.16),

$$K_9 = \frac{2^{2d+1} I_2}{c_d I_1}.$$

We shall prove that

$$\int_F \frac{h^+(T)}{|T - X_n|^d (|T|^2 + y^2)^{\frac{d}{2}}} dT < \frac{K_8}{2K_9} \cdot \frac{1}{m(n, y)^d}. \quad (5.5.15)$$

Consider

$$Q = \int_{|T| < 1} \frac{m(n, y)^d h^+(T)}{|T - X_n|^d (|T|^2 + y^2)^{\frac{d}{2}}} dT.$$

Suppose that $|X_n| > y$. Then $m(n, y) = |X_n|$. If $0 < |T| \leq \frac{1}{2}|X_n|$, then $|T - X_n| \geq \frac{1}{2}|X_n| = \frac{1}{2}m(n, y)$. If $\frac{1}{2}|X_n| < |T| < 1$, then $(|T|^2 + y^2)^{\frac{1}{2}} \geq \frac{1}{2}|X_n| = \frac{1}{2}m(n, y)$ and so

$$\begin{aligned} Q &\leq 2^d \left\{ \int_{0 < |T| \leq \frac{|X_n|}{2}} \frac{h^+(T) dT}{(|T|^2 + y^2)^{\frac{d}{2}}} + \int_{\frac{|X_n|}{2} < |T| < 1} \frac{h^+(T)}{|T - X_n|^d} dT \right\} \\ &\leq 2^d \left\{ \int_{|T| < 1} \frac{h^+(T)}{|T|^d} dT + \int_{|T| < 1} \frac{h^+(T)}{|T - X_n|^d} dT \right\}. \end{aligned} \quad (5.5.16)$$

Suppose that $|X_n| \leq y$. Then $m(n, y) = y$ and $(|T|^2 + y^2)^{\frac{1}{2}} \geq y$, so

$$Q \leq \int_{|T| < 1} \frac{h^+(T)}{|T - X_n|^d} dT,$$

and (5.5.16) holds in general. We know from Lemma 5.6 (iii) and (5.4.3) that

$$\begin{aligned} \int_{|T| < 1} \frac{h^+(T)}{|T - X_n|^d} dT &\leq 2^{d+1} \int_{|T| < 1} \frac{h^+(T)}{|T|^d} dT \\ &\leq 2^{d+1} \frac{I_1}{8.48^d} \\ &= \frac{1}{2^{d+1}} \frac{K_8}{2K_9}. \end{aligned}$$

Thus (5.5.16) yields

$$Q \leq 2^{d+1} \left(\frac{1}{2^{d+1}} \frac{K_8}{2K_9} \right) = \frac{K_8}{2K_9}.$$

This proves (5.5.15).

Thus, (5.5.13), (5.5.14) and (5.5.15) yield

$$\begin{aligned}
\frac{U(0, y)}{y} &= \frac{u_1(0, y) - u_2(0, y)}{y} \\
&\geq \frac{K_8}{2} \sum_{n=1}^{\infty} \frac{\mu_n r_n^{d-1}}{m(n, y)^d} \\
&\geq \frac{K_8}{2} \sum_{n=1}^{n_y-1} \frac{\mu_n r_n^{d-1}}{|X_n|^d} \\
&= \frac{K_8}{2K_1} \left(\frac{\mu_{n_y}}{r_{n_y}} \right),
\end{aligned}$$

by (5.4.4) where n_y is the largest n for which $|X_{n_y}| \geq y$. Thus n_y tends to infinity as y tends to zero and, since $\mu_n/r_n \rightarrow \infty$ as $n \rightarrow \infty$ by (5.4.6) and (5.3.13), (5.5.12) follows. This completes the proof of Lemma 5.10.

Chapter 6

Further results on positive harmonic functions in Lipschitz domains

6.1 Statement of results

We deal in this chapter with two results which complement Theorem 4.1. In that theorem we dealt with the more difficult case when the integral in (4.4.2) is finite and that in (4.4.3) is infinite. We will deduce from Theorem 4.1 a corresponding result when the integral in (4.4.3) is finite and that in (4.4.2) is infinite.

We work in \mathcal{R}^d where $d \geq 3$. The results in the plane are known. As before, $h(X)$ is a real-valued Lip 1 function with Lipschitz constant 1 which vanishes outside $|X| < 1$ and for which $h(0) = 0$. We put

$$D = \{(X, y) \in \mathcal{R}^d : y > h(X)\} \quad (6.1.1)$$

and, for positive ϵ , write D_ϵ for the component of $D \cap B(0, \epsilon)$ which contains those points $(0, y)$ with $0 < y < \epsilon$.

Theorem 6.1 *Suppose that the integral*

$$\int_{|T|<1} \frac{h^+(T)}{|T|^d} dT = \infty \quad (6.1.2)$$

and that

$$\int_{|T|<1} \frac{h^-(T)}{|T|^d} dT < \infty. \quad (6.1.3)$$

If ϵ is positive and u is any positive harmonic function in D_ϵ which is continuous in \overline{D}_ϵ and vanishes at all points of $\partial D \cap \partial D_\epsilon$, then

$$\frac{u(0, y)}{y} \rightarrow 0 \text{ as } y \rightarrow 0^+.$$

The case when both integrals are finite remains. We prove

Theorem 6.2 *Suppose that both*

$$\int_{|T|<1} \frac{h^+(T)}{|T|^d} dT < \infty \quad (6.1.4)$$

and that

$$\int_{|T|<1} \frac{h^-(T)}{|T|^d} dT < \infty. \quad (6.1.5)$$

If ϵ is positive and if u is any positive harmonic function in D_ϵ which is continuous in \overline{D}_ϵ and vanishes at all points of $\partial D \cap \partial D_\epsilon$, then

$$l = \lim_{y \rightarrow 0} \frac{u(0, y)}{y}$$

exists and $0 < l < \infty$.

Both Theorem 6.1 and Theorem 6.2 were proved by Burdzy (see [8], Theorem 4.2) using probabilistic methods. Once again our proofs are classical.

6.2 Proof of Theorem 6.1

The proof of Theorem 6.1 is based on methods taken from [18] and [29] and uses the theory described in Sections 1.2.7 and 1.2.8.

Let a positive ϵ be given. We let D^+ and D^- be the components of

$$\{(X, y) \in B(0, \epsilon) : y > h(X)\}$$

and

$$\{(X, y) \in B(0, \epsilon) : y < h(X)\}$$

containing $(0, \epsilon/2)$ and $(0, -\epsilon/2)$ respectively. Let E^+ and E^- denote those parts of the boundaries of D^+ and D^- on $S(0, \epsilon)$ respectively.

Define ω^+ and ω^- on $B(0, \epsilon)$ as follows. Let

$$\omega^+(X, y) = \begin{cases} \omega((X, y), E^+; D^+), & (X, y) \in D^+, \\ 0, & (X, y) \notin D^+, \end{cases}$$

and

$$\omega^-(X, y) = \begin{cases} \omega((X, y), E^-; D^-), & (X, y) \in D^-, \\ 0, & (X, y) \notin D^-, \end{cases}$$

where ω denotes harmonic measure. Then ω^+ and ω^- are subharmonic and nonnegative in $B(0, \epsilon)$. In what follows we work with ω^+ but by symmetry the same analysis applies equally well to ω^- .

For each r with $0 < r < \epsilon$ we make the following definitions. Let $E^+(r)$ denote the intersection of the set $\omega^+ > 0$ with $S(0, r)$ and let $\alpha^+(r)$ be the characteristic constant of $E^+(r)$. As in (1.2.12), put

$$m^+(r) = \left\{ \frac{1}{c_d r^{d-1}} \int_{E^+(r)} \omega^+(X, y)^2 d\sigma \right\}^{\frac{1}{2}},$$

so that $m^+(r)$ denotes the Carleman mean of ω^+ .

Theorem 1.F then gives that

$$r \frac{d}{dr} \{ \log A^+(r) \} \geq 2\alpha^+(r) + d - 2, \quad (6.2.1)$$

where

$$A^+(r) = r \frac{d}{dr} (m^+(r)^2 r^{d-2}). \quad (6.2.2)$$

Suppose that $0 < r < \epsilon/2$. Integrating (6.2.1) yields

$$\log A^+(t) \Big|_r^{\epsilon/2} \geq 2 \int_r^{\epsilon/2} \frac{\alpha^+(t)}{t} dt + (d-2) \log \left(\frac{\epsilon}{2r} \right).$$

Thus

$$\frac{A^+(\epsilon/2)}{A^+(r)} \geq \left(\frac{\epsilon}{2r} \right)^{d-2} \exp \left\{ 2 \int_r^{\epsilon/2} \frac{\alpha^+(t)}{t} dt \right\},$$

giving

$$\frac{A^+(r)}{r^d} \leq A^+(\epsilon/2) \left(\frac{2}{\epsilon} \right)^{d-2} r^{-2} \exp \left\{ -2 \int_r^{\epsilon/2} \frac{\alpha^+(t)}{t} dt \right\}. \quad (6.2.3)$$

Now

$$\frac{A^+(r)}{r^d} = (d-2) \left(\frac{m^+(r)}{r} \right)^2 + \frac{2m^+(r)}{r} \frac{d}{dr} (m^+(r))$$

and so, since $(m^+)'(r) \geq 0$,

$$\frac{A^+(r)}{r^d} \geq (d-2) \left(\frac{m^+(r)}{r} \right)^2. \quad (6.2.4)$$

From (6.2.3) and (6.2.4) we obtain

$$m^+(r) \leq \left(\frac{A^+(\epsilon/2)}{d-2} \right)^{\frac{1}{2}} \left(\frac{2}{\epsilon} \right)^{\frac{d-2}{2}} \exp \left\{ - \int_r^{\frac{\epsilon}{2}} \frac{\alpha^+(t)}{t} dt \right\}.$$

With $E^-(r), \alpha^-(r), m^-(r)$ and $A^-(r)$ defined analogously, we obtain

$$m^-(r) \leq \left(\frac{A^-(\epsilon/2)}{d-2} \right)^{\frac{1}{2}} \left(\frac{2}{\epsilon} \right)^{\frac{d-2}{2}} \exp \left\{ - \int_r^{\frac{\epsilon}{2}} \frac{\alpha^-(t)}{t} dt \right\}.$$

Thus

$$m^+(r)m^-(r) \leq C \exp \left\{ - \int_r^{\frac{\epsilon}{2}} \frac{\alpha^+(t) + \alpha^-(t)}{t} dt \right\} \quad (6.2.5)$$

where C depends only on ϵ , the function $h(T)$ and d but not on r .

Let $c_d r^{d-1} S^+(r)$ and $c_d r^{d-1} S^-(r)$ denote the areas of $E^+(r)$ and $E^-(r)$ respectively. Thus

$$S^+(r) + S^-(r) \leq 1.$$

Now from Theorem 1.D and Theorem 1.E we have that

$$\alpha^+(r) \geq 2(1 - S^+(r))$$

and that

$$\alpha^-(r) \geq 2(1 - S^-(r)),$$

so that

$$\alpha^+(r) + \alpha^-(r) \geq 2(2 - S^+(r) - S^-(r)) \geq 2.$$

Thus, by (6.2.5),

$$\begin{aligned} m^+(r)m^-(r) &\leq C \exp \left\{ - \int_r^{\frac{\epsilon}{2}} \frac{2}{t} dt \right\} \\ &= \frac{4C}{\epsilon^2} r^2. \end{aligned} \quad (6.2.6)$$

Note by (1.2.15) that

$$\omega^+ \left(0, \frac{r}{2} \right) \leq 3 \cdot 2^{d-2} m^+(r)$$

and that

$$\omega^- \left(0, \frac{r}{2} \right) \leq 3 \cdot 2^{d-2} m^-(r).$$

Thus by (6.2.6)

$$\left(\frac{\omega^+(0, r)}{r} \right) \left(\frac{\omega^-(0, r)}{r} \right) \leq C. \quad (6.2.7)$$

Note that in the above analysis required to establish (6.2.7) no use was made of the fact that the function $h(T)$ is Lipschitz. Thus (6.2.7) holds for any continuous function $h(T)$.

In our situation, the assumptions that $h(T)$ is Lipschitz and that (6.1.2) and (6.1.3) hold enable us to deduce from Theorem 4.1 that

$$\frac{\omega^-(0, r)}{r} \rightarrow \infty$$

as $r \rightarrow 0$. The inequality (6.2.7) then yields

$$\lim_{r \rightarrow 0} \frac{\omega^+(0, r)}{r} = 0,$$

which completes the proof of Theorem 6.1 by Theorem 5.A.

6.3 Proof of Theorem 6.2

The assumptions (6.1.4) and (6.1.5) imply that the boundary of D near 0 is almost flat, so that small neighbourhoods of 0 look like half-spheres. Thus u in D is not very different from a positive harmonic function in a half-sphere for which the limit in question certainly exists. The strategy of the proof is to make the above observations precise. Before embarking on the proof proper we make a

Definition 6.1 *Suppose that D is a domain in \mathcal{R}^d and that E is a Borel subset of the boundary of D . Suppose further that u is a continuous real-valued function on E . Then we write*

$$\phi(x; E, u, D)$$

for the harmonic function in D with boundary values of $u(x)$ on E and 0 otherwise.

For r in $(0, \epsilon)$, we let D_r and D_r^+ be the domains which are the intersection of $B(0, r)$ with $y > h(X)$ and $y > h^+(X)$ respectively. Let C_r denote that part of the boundary of D_r^+ on $S(0, r)$ and let F_r denote the intersection of D_r with the hyperplane $y = 0$.

Now fix r_0 in $(0, \epsilon)$. For x in $D_{r_0}^+$, we have

$$u(x) = \phi(x; C_{r_0}, u, D_{r_0}^+) + \phi(x; F_{r_0}, u, D_{r_0}^+). \quad (6.3.1)$$

Next, for $0 < \rho < r_0$, let S_ρ denote the half-sphere which is the intersection of $B(0, \rho)$ and H . Also, let H_ρ denote all those (X, y) in S_ρ which lie on the graph $(X, h^+(X))$.

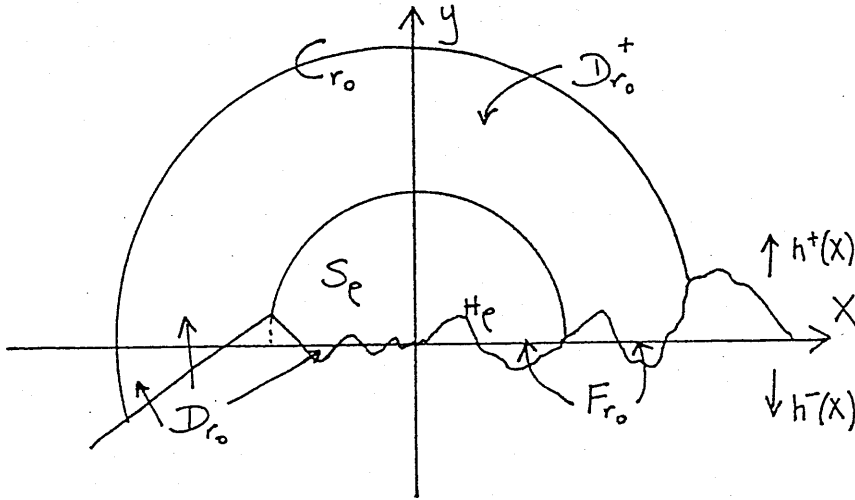


Figure 6.1 Notation for (6.3.1) and (6.3.2)

Then, writing $u_0(x) = \phi(x; C_{r_0}, u, D_{r_0}^+)$, we have for x in D_ρ^+

$$u_0(x) = \phi(x; C_\rho, u_0, S_\rho) - \phi(x; H_\rho, \phi(x; C_\rho, u_0, S_\rho), D_\rho^+). \quad (6.3.2)$$

We write, for $x \in D_\rho^+$,

$$v_\rho(x) = \phi(x; C_\rho, u_0, S_\rho),$$

and so we have, by (6.3.1) and (6.3.2) that, for x in D_ρ^+ ,

$$u(x) = v_\rho(x) + \phi(x; F_{r_0}, u, D_{r_0}^+) - \phi(x; H_\rho, v_\rho, D_\rho^+). \quad (6.3.3)$$

Thus, by (6.3.3), for $0 < \rho < r_0$ and $0 < y < \rho$,

$$\frac{v_\rho(0, y)}{y} - \frac{\phi((0, y); H_\rho, v_\rho, D_\rho^+)}{y} \leq \frac{u(0, y)}{y} \leq \frac{v_\rho(0, y)}{y} + \frac{\phi((0, y); F_{r_0}, u, D_{r_0}^+)}{y}. \quad (6.3.4)$$

Now

$$\lim_{y \rightarrow 0} \frac{v_\rho(0, y)}{y}$$

exists and equals K_ρ , say. Thus, by (6.3.4),

$$\begin{aligned} \limsup_{y \rightarrow 0} \frac{u(0, y)}{y} &\leq \limsup_{y \rightarrow 0} \left(\frac{v_\rho(0, y)}{y} + \frac{\phi((0, y); F_{r_0}, u, D_{r_0}^+)}{y} \right) \\ &\leq \limsup_{y \rightarrow 0} \frac{v_\rho(0, y)}{y} + \limsup_{y \rightarrow 0} \frac{\phi((0, y); F_{r_0}, u, D_{r_0}^+)}{y} \\ &= K_\rho + \limsup_{y \rightarrow 0} \frac{\phi((0, y); F_{r_0}, u, D_{r_0}^+)}{y} \end{aligned} \quad (6.3.5)$$

and

$$\begin{aligned} \liminf_{y \rightarrow 0} \frac{u(0, y)}{y} &\geq \liminf_{y \rightarrow 0} \left(\frac{v_\rho(0, y)}{y} - \frac{\phi((0, y); H_\rho, v_\rho, D_\rho^+)}{y} \right) \\ &\geq \liminf_{y \rightarrow 0} \frac{v_\rho(0, y)}{y} - \limsup_{y \rightarrow 0} \frac{\phi((0, y); H_\rho, v_\rho, D_\rho^+)}{y} \\ &= K_\rho - \limsup_{y \rightarrow 0} \frac{\phi((0, y); H_\rho, v_\rho, D_\rho^+)}{y}. \end{aligned} \quad (6.3.6)$$

Let a positive δ be given. We now show that for all sufficiently small r_0 ,

$$\limsup_{y \rightarrow 0} \frac{\phi((0, y); F_{r_0}, u, D_{r_0}^+)}{y} < \delta \quad (6.3.7)$$

and that, if r_0 is fixed, then for all sufficiently small ρ ,

$$\limsup_{y \rightarrow 0} \frac{\phi((0, y); H_\rho, v_\rho, D_\rho^+)}{y} < \delta. \quad (6.3.8)$$

It then follows from (6.3.5) and (6.3.6) that

$$\limsup_{y \rightarrow 0} \frac{u(0, y)}{y} - \liminf_{y \rightarrow 0} \frac{u(0, y)}{y} < 2\delta$$

so that

$$\lim_{y \rightarrow 0} \frac{u(0, y)}{y}$$

exists.

To complete the proof of the theorem we need to establish (6.3.7) and (6.3.8). We begin with (6.3.7).

Note that for each r in $(0, \epsilon)$,

$$\phi(x; F_r, u, D_r^+) \leq \phi(x; F_r, u, S_\epsilon) \quad (6.3.9)$$

since D_r^+ is a subdomain of S_ϵ . Moreover,

$$\lim_{y \rightarrow 0} \frac{\phi((0, y); F_1, u, S_\epsilon)}{y}$$

exists since S_ϵ is a half-sphere, though the limit might possibly be infinite. Let D_ϵ^- be the domain which is the intersection of $B(0, \epsilon)$ with $y > h^-(X)$. We can now obtain from (6.2.3) a lemma analogous to Lemma 5.5 which when combined with the proof of Theorem Part (i) in [29] yields

$$\limsup_{y \rightarrow 0} \frac{\phi(x; C_\epsilon, u, D_\epsilon^-)}{y} < \infty.$$

Since $\phi(x; F_\epsilon, u, S_\epsilon) \leq \phi(x; C_\epsilon, u, D_\epsilon^-)$ we deduce that

$$\lim_{y \rightarrow 0} \frac{\phi(x; F_\epsilon, u, S_\epsilon)}{y} \leq \infty.$$

Thus

$$\lim_{r \rightarrow 0} \left(\lim_{y \rightarrow 0} \frac{\phi(x; F_r, u, S_\epsilon)}{y} \right) = 0$$

and so, by (6.3.9), (6.3.7) holds.

It remains to establish (6.3.8). This is a little more involved than (6.3.7).

Note that if $0 < \rho_1 < \rho_2 \leq r_0$ then, for x in S_{ρ_1} ,

$$v_{\rho_1}(x) \leq v_{\rho_2}(x)$$

since on C_{ρ_1}

$$u_0(x) \leq v_{\rho_2}(x).$$

It then follows that

$$\begin{aligned} \phi(x; H_\rho, v_\rho, D_\rho^+) &\leq \phi(x; H_\rho, v_{r_0}, D_\rho^+) \\ &\leq \phi(x; H_\rho, v_{r_0}, \Omega_\rho) \end{aligned} \quad (6.3.10)$$

where

$$\Omega_\rho = H \setminus \{(X, y) \text{ in } B(0, \rho) \text{ with } y < h^+(X)\}$$

We now construct a harmonic majorant for $\phi(x; H_\rho, v_{r_0}, \Omega_\rho)$ by extending into the half-space H a suitable function $\omega(T)$ on the boundary of H .

First of all, since v_{r_0} is the restriction to S_{r_0} of a harmonic function in $B(0, r_0)$ we have that

$$\left| \frac{\partial v_{r_0}}{\partial y} \right| \leq \frac{C}{r_0} A(\lambda)$$

when $|x| < \lambda r_0$, $0 < \lambda < 1$ and where C depends only on the maximum of u and on d . Thus, taking $\lambda = \frac{3}{4}$ say,

$$v_{r_0}(X, y) \leq \frac{C_1 y}{r_0} \quad (6.3.11)$$

when $|X| < r_0/2$ and $|(X, y)| < 3r_0/4$. Clearly, we can then increase C_1 if necessary so that (6.3.11) holds for $|X| < r_0/2$ and any y so that $|(X, y)| < r_0$.

Next we define the boundary function $\omega(T)$. Define

$$\omega(T) = C_2 \max \left\{ h^+(X) : |X - T| < \frac{h^+(X)}{2} \right\}, \quad (6.3.12)$$

where

$$C_2 = \frac{\pi C_1 (d-1) 2^{d-1}}{r_0 \sigma_{d-1}} \cdot \left(\frac{5}{4} \right)^{\frac{d}{2}}.$$

We assert that

$$\omega(T) < 2C_2 h^+(T). \quad (6.3.13)$$

In fact,

$$h^+(X) \leq |X - T| + h^+(T)$$

because h is Lipschitz with constant 1 and so, if $|X - T| < h^+(X)/2$, then

$$h^+(X) < 2h^+(T).$$

Next, the harmonic extension ω of $\omega(T)$ into H majorises $\phi(x; H_\rho, v_{r_0}, \Omega_\rho)$. For this it is enough by (6.3.11), that if $|X| < \rho$ then

$$\omega(X, h^+(X)) \geq \frac{C_1 h^+(X)}{r_0}. \quad (6.3.14)$$

We now estimate $\omega(X, h^+(X))$ from below.

$$\begin{aligned}\omega(X, h^+(X)) &= \frac{1}{\pi} \int_F \frac{h^+(X)\omega(T)}{[|T-X|^2 + h^+(X)^2]^{\frac{d}{2}}} dT \\ &\geq \frac{h^+(X)}{\pi} \int_{|X-T| < \frac{h^+(X)}{2}} \frac{\omega(T)}{[|T-X|^2 + h^+(X)^2]^{\frac{d}{2}}} dT.\end{aligned}$$

By definition of $\omega(T)$, we have

$$\omega(T) \geq C_2 h^+(X) \text{ when } |T-X| < \frac{h^+(X)}{2}.$$

Furthermore, when $|T-X| < h^+(X)/2$,

$$|T-X|^2 + h^+(X)^2 \leq h^+(X)^2 \left(\frac{1}{4} + 1\right) = \frac{5}{4} h^+(X)^2.$$

Thus,

$$\begin{aligned}\omega(X, h^+(X)) &\geq \frac{h^+(X)}{\pi} \cdot C_2 h^+(X) \cdot \left(\frac{4}{5h^+(X)^2}\right)^{\frac{d}{2}} \cdot \left(\frac{\sigma_{d-1}}{d-1}\right) \cdot \left(\frac{h^+(X)}{2}\right)^{d-1} \\ &= \frac{C_2}{\pi} \left(\frac{4}{5}\right)^{\frac{d}{2}} \left(\frac{\sigma_{d-1}}{d-1}\right) \frac{1}{2^{d-1}} h^+(X) \\ &= \frac{C_1}{r_0} h^+(X).\end{aligned}$$

Lastly, note that by (6.3.10), (6.3.13) and (6.3.14)

$$\begin{aligned}\limsup_{y \rightarrow 0} \frac{\phi((0, y); H_\rho, v_\rho, D_\rho^+)}{y} &\leq \limsup_{y \rightarrow 0} \frac{\phi((0, y); H_\rho, v_{r_0}, \Omega_\rho)}{y} \\ &\leq \limsup_{y \rightarrow 0} \frac{\omega(0, y)}{y} \\ &= \frac{1}{\pi} \int_F \frac{\omega(T)}{|T|^d} dT \\ &\leq \frac{2C_2}{\pi} \int_{|T| < \rho} \frac{h^+(T)}{|T|^d} dT.\end{aligned}$$

Since, if ρ is small,

$$\int_{|T| < \rho} \frac{h^+(T)}{|T|^d} dT < \delta,$$

this establishes (6.3.8) and completes the proof.

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